Applied Partial Differential Equations

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目录

1 Ordinary Differential Equation

1.1 Modelling

1. Free Falling Law
   $F = ma, a = \ddot{x} = \frac{d^2x}{dt^2}, m\ddot{x} = -mg, \ddot{x} = -g, \Rightarrow x(t) = \frac{1}{2}gt^2 + C_0 + C_1 t, x(0) = C_0, \dot{x}(0) = C_1.$

2. Population Model (Malthus)
   $x(t) = \#$ of population at $t, \frac{\dot{x}(t)}{x(t)} =$ relative growth rate $= \alpha > 0$. Solu: $x(t) = x(0)e^{\alpha t}$.
   Modification: Logistic Model
   illed $x(t)$,healthy $y(t)$,$x(t) + y(t) = N, \frac{\dot{x}(t)}{x(t)} = \alpha y = \alpha(N - x)$, denote $\bar{x} = \frac{x}{N}$, we have
   $\dot{\bar{x}} = \beta\bar{x}(1 - \bar{x}), \bar{x}(t) = \frac{\bar{x}(0)e^{\beta t}}{1+\bar{x}(0)(e^{\beta t}-1)}$.

3. Harmonic Oscillator
   $F = ma, x(t) =$ displacement of t, $F = -kx$(Hooke's Law), then $m\ddot{x} = -kx, \ddot{x} + \omega^2 x = 0$.
   Solu: $x(t) = e^{\pm i\omega t}$.

4. Free Fall at Large Distance
   $m\ddot{x} = F = -\frac{GMm}{x^2}, \ddot{x} = -\frac{GM}{x^2}, x(0) = R, \dot{x}(0) = 0.$
   ANSATZ: $x(t) = at^b$, we have $ab(b-1)t^{b-2} = -GMa^{-2}t^{-2b}, b = \frac{2}{3}, a = (\frac{9GM}{2})^{\frac{1}{3}}$.
   We find that initial conditions do not match, thus modifying solution as $a(t \pm c)^b$, etc.

1.2 Matrix Series

$\begin{cases} \dot{x} = Ax \\ x|_{t=0} = x_0 \end{cases}$ Theorem: $\exists\ 1\ x(t) = e^{At}x_0$, where $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$

Generally speaking, to compute e^{At}, there are 3 methods as follows.

1. $PAP^{-1} = \Lambda$

2. regulation. For example, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = -I$.

3. putzer method.
   $e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t)P_j := \Phi(t)$, where
   $P_0 = I \qquad P_j = (A - \lambda_j I)P_{j-1} \qquad \begin{cases} \dot{r}_1(t) = \lambda_1 r_1(t) \\ r_1(0) = 1 \end{cases} \quad \begin{cases} \dot{r}_j(t) = \lambda_j r_j(t) + r_{j-1}(t) \\ r_j(0) = 0 \end{cases}$
   Proof: $\dot{\Phi}(t) = \sum_{j=1}^{n} \dot{r}_j(t)P_{j-1} = \sum_{j=1}^{n}(\lambda_j r_j(t) + r_{j-1}(t))P_{j-1} = \sum_{j=1}^{n} \lambda_j r_j(t)P_{j-1} + \sum_{j=0}^{n-1} r_j(t)(A - \lambda_j I)P_{j-1} = A\Phi(t)$. The last step: $P_n = 0$ (Hamilton-Cayley Theorem), thus $AP_{n-1} = \lambda_n P_{n-1}$.

1.3 Matrix Norm and Periodic Coefficient Matrix

Vector Norm: $||x||_p = (\sum |x_i|^p)^{\frac{1}{p}}$. Operator norm: $||A||_p = \max_{||x||_p \neq 0} \frac{||Ax||_p}{||x||_p} = \max_{||x||_p=1} ||Ax||_p$.

For example, $||A||_1 = \max_{||x||_1=1} ||Ax||_1 = \max(|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|), i.e. ||A||_1 =$ maximum of column $(\max_i \sum_j |A_{ji}|)$, and $||A||_\infty =$ maximum of row $(\max_i \sum_j |A_{ij}|)$.

Prop: $||AB|| \le ||A|| \cdot ||B||, ||\sum \frac{A^k}{k!}|| \le e^{||A||} < \infty, AB = BA \Leftrightarrow e^{(A+B)t} = e^{At} \cdot e^{Bt} = e^{Bt} \cdot e^{At}, \forall t$

Proof: $\begin{cases} \dot{x}_t = (A+B)x \\ x|_{t=0} = x_0 \end{cases}$. Use the uniqueness of the solution, where $e^{(A+B)t}$ and $e^{At} \cdot e^{Bt}$ are both solutions.

Consider $\begin{cases} \dot{x}_t = A(t)x \\ x|_{t=0} = x_0 \end{cases}$. Denote $x(t) = \Phi(t)x_0$(fundamental matrix), then $\begin{cases} \dot{\Phi}_t = A(t)\Phi(t) \\ \Phi(0) = I \end{cases}$.

Claim: $\exists$ 1 solution.

Liouwille's Th: det $\Phi(t) =$ det $\Phi(0) \cdot e^{\int_0^t trA(s)ds}$.

In fact, $\forall Z(t)$ is invertible, det $Z(t) =$ det $Z(0) \cdot e^{\int_0^t tr(Z^{-1}(s)\dot{Z}(s))ds}$.

We note that even if $A(t+T) = A(t)$, $x(t)$ or $\Phi(t)$ can be not periodic. However, we have

Floquet/Bloch Thm: $\Phi(t+T) = \Phi(t)Q$ if $A(t+T) = A(t)$ where Q is a constant matrix.

Proof: $\frac{d}{dt}(\Phi^{-1}(t)\Phi(t+T)) = -\Phi^{-1}(t)\dot{\Phi}(t)\Phi^{-1}(t)\Phi(t+T) + \Phi^{-1}(t)\dot{\Phi}(t+T) = -\Phi^{-1}(t)A(t)\Phi(t+T) + \Phi^{-1}(t)A(t+T)\Phi(t+T) = 0$.

Prop: det $Q \neq 0$, $Q = e^{TB}$ where $B \in C^{n\times n}$. And if denoting $\Psi(t) = \Phi(t)e^{-tB}$, we have $\Psi(t+T) = \Psi(t)$. To go further, by denoting $x(t) = \Psi(t)z(t)$ and calculating, we have $\dot{z} = Bz$.

Proof: $x(t) = \Psi(t)z(t)$, hence $\dot{x} = \dot{\Psi}(t)z(t) + \Psi(t)\dot{z}(t) = A(t)\Psi(t)z(t), \dot{\Phi}(t)e^{-tB}z(t) - \Phi(t)e^{-tB}B$ $z(t) + \Psi(t)\dot{z}(t) = A(t)\Phi(t)e^{-tB}z(t), \dot{\Phi}(t) = A(t)\Phi(t), \dot{z}(t) = Bz(t)$.

We must point out that this conclusion has only theoratical value, because it uses the equation $Q = e^{TB}$ where matirx B is very difficult to compute.

Example: $A(t) = \begin{pmatrix} 1 & 0 \\ \cos t & 0 \end{pmatrix}$. Obviously $A(t)$ is periodic and the fundamental matrix $\Phi(t)$ is $\begin{pmatrix} e^t & 0 \\ \int_0^t e^s\cos s ds & 1 \end{pmatrix}$. Here $Q = \Phi^{-1}(t)\Phi(t+2\pi) = \begin{pmatrix} e^{2\pi} & 0 \\ \int_0^{t+2\pi} e^s\cos s ds - e^{2\pi}\int_0^t e^s\cos s ds & 1 \end{pmatrix}$. It seems $Q = Q(t)$, but if you compute $\frac{dQ}{dt}$ you will find it a constant.

D'Alambert's reduction (not required): for differential equation $\dot{x} = A(t)x$ where $A \in R^{n\times n}$, if one solution $y(t)$ is known, we can reduce the dimension of solution space from n to $n-1$.

ANSATZ: $x(t) = \phi(t)y(t) + z(t)$. By computing we get the result: $\dot{\phi}(t) = \sum_{j=2}^n A_{1j}z_j/y_1(t)$,$\dot{z}_i(t) = \sum_{j=2}^n (A_{ij} - A_{1j}\frac{y_i}{y_1})z_j$ where $i = 2, \ldots, n$.

1.4 Duhammel's Principle (Variation of Constants)

Consider differential equations $\begin{cases} \dot{x} = Ax + f \\ x|_{t=0} = x_0 \end{cases}$ and $\begin{cases} \dot{x} = A(t)x + f(t) \\ x|_{t=0} = x_0 \end{cases}$.

ANSATZ: For LHS, $x(t) = e^{At}C(t), \dot{x} = Ax + e^{At}\dot{C}(t) = Ax + f(t), \dot{C}(t) = e^{-At}f(t)$, hence $x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds$. For RHS, we denote $\Phi(t)$ as fundamental matrix, then $x(t) = \Phi(t)C(t), \dot{x} = \dot{\Phi}(t)C(t) + \Phi(t)\dot{C}(t), \dot{C}(t) = \Phi^{-1}(t)f(t)$.

Example: $\begin{cases} \dot{Z} = AZ + ZB \\ Z(0) = Q \end{cases}$. Assume $Z(t) = e^{At}Y(t)$, hence $\dot{Z} = AZ + e^{At}Y(t)B$, so we have $\begin{cases} \dot{Y}(t) = Y(t)B \\ Y(0) = Q \end{cases}$, $Y(t) = Qe^{Bt}, Z(t) = e^{At}Qe^{Bt}$.

High-order ODE $\to$ 1st ODEs: $\begin{cases} \ddot{x}+\omega^2 x = f(t) \\ x(0)=x_0, \dot{x}(0)=x_1 \end{cases}$. By denoting $Z=(x,\dot{x}/\omega)^T$, we have $\dot{Z}=AZ+F$ where $A=\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$,$F=(0,f(t)/\omega)^T$. The solution is $Z(t)=\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} x_0 \\ \frac{x_1}{\omega} \end{pmatrix}$ $+\int_0^t \begin{pmatrix} \cos[\omega(t-s)] & \sin[\omega(t-s)] \\ -\sin[\omega(t-s)] & \cos[\omega(t-s)] \end{pmatrix}\begin{pmatrix} 0 \\ \frac{f(s)}{\omega} \end{pmatrix} ds$. If $x_0=x_1=0$ and $f(t)=e^{i\nu t}$, $x(t)=\frac{1}{\omega}\int_0^t \sin[\omega(t-s)]f(s)ds \sim \int_0^t e^{i\omega(t-s)}e^{i\nu s}ds = e^{i\omega t}\int_0^t e^{i(\nu-\omega)s}ds$. We find $\omega\to\nu$, resonance forms.

Consider $x^{(n)}+a_1x^{(n-1)}+\cdots+a_{n-1}x'+a_nx=0(f(t))$. By denoting $Z=(x,\dot{x},\ddot{x},\ldots,x^{(n-1)})^T$, we have $\dot{Z}=AZ+F$ where $F=(0,\ldots,0,f)^T$. BTW, if $a_i\to a_i(t)$, we have $\dot{Z}=A(t)Z(t)+F(t)$. Assume $x=e^{\lambda t}$, then $\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n=0$, e.g. $\det(\lambda I-A)=0$. Hence the fundamental solutions are $\{e^{\lambda_i t}\}_{i=1}^n$. If $\lambda_1=\lambda_2$, they become $\{e^{\lambda t},\ldots,t^{n-1}e^{\lambda t}\}$.

Euler equation: $a_i=t^i\frac{d^ix}{dt^i}$, e.g. $t^2\ddot{x}+t\dot{x}+x=0$. Via denoting $t=e^s$, we can get $x(t)=x(e^s)=y(s)$ and $t\frac{dx}{dt}=\frac{dy}{ds}$, thus initial question becomes an ode of constant coefficients. To go further, the solution has the form $y(s)=e^{\lambda s}$ while $t=e^s$, hence $x(t)=t^\lambda$. More specifically, considering the example, we have $\lambda^2+1=0,\lambda=\pm i$.

1.5 Existence and Uniqueness of Solution

$\begin{cases} \dot{x}=f(t,x) \\ x|_{t=0}=x_0 \end{cases}$, Lip: $||f(t,x_1)-f(t,x_2)||\le L||x_1-x_2||$. A special example: $f(t,x)=A(t)x$.

Granwall's inequality: $x(t)=x_0+\int_0^t f(s,x(s))ds, ||x_1(t)-x_2(t)||\le L\int_0^t ||x_1(s)-x_2(s)||ds := LF(t)$, we claim $F(t)\le F(0)e^{tL}$ because $\frac{\dot{F}(t)}{F(t)}\le L$. To dig a little deeper, if $f(t)\le A+\int_0^t b(s)f(s)ds$ with $f(t)\ge 0, A\ge 0, b(s)\ge 0$, we have $f(t)\le Ae^{\int_0^t b(s)ds}$.

Proof: denoting $F(t)=A+\int_0^t b(s)f(s)ds$, we have $\frac{\dot{F}(t)}{F(t)}\le b(t), F(t)\le F(0)e^{\int_0^t b(s)ds}$.

Uniqueness: $z=x_1-x_2$, then $\frac{1}{2}\frac{d}{dt}||z||^2=(z,\dot{z})=(z,f(t,x_1)-f(t,x_2))\le ||z||\cdot||f(t,x_1)-f(t,x_2)||\le ||z||\cdot L||x_1-x_2||=L||z||^2$, i.e. $\frac{\frac{d}{dt}||z||^2}{||z||^2}\le 2L$. $||z(t_0)||=0$ and $||z||\ge 0$, thus $||z||\equiv 0$.

Existence: $\begin{cases} \dot{x}=f(t,x) \\ x|_{t=0}=x_0 \end{cases}$ $\{(t,x)|\,|t-t_0|\le a, |x-x_0|\le b\}$. If 1) $|f(t,x)|\le M$, 2) $|\frac{\partial f}{\partial x}|\le L\Rightarrow$ $f(t,x)$ is Lip w.r.t. x, then $\exists\,1\,x$ s.t. $\begin{cases} \dot{x}=f(t,x) \\ x|_{t=0}=x_0 \end{cases}$ $\to$ classical solu. $x(t)=x_0+\int_{t_0}^t f(s,x(s))ds\to$ strong solu, $\int_I x(t)\phi(t)dt=x_0\int_I\phi(t)dt+\int_I\int_{t_0}^t f(s,x(s))ds\phi(t)dt\,\forall\,\phi(t)\to$ weak solu.

Lemma: classical solu, strong solu, weak solu are essentially equivalent.

Proof: Picard iteration. $x_n=x_0+\int_{t_0}^t f(s,x_{n-1}(s))ds$ for $n=1,2,\cdots$.Then $|x_1-x_0|\le M(t-t_0), |x_2-x_1|\le\frac{LM}{2}(t-t_0)^2,\cdots,\Rightarrow\sum|(x_n-x_{n-1})(t)|\le\sum\frac{L^{n-1}M}{n!}(t-t_0)^n, x_n(t)\to x_\infty(t)$. Usually, $x_\infty(t)$ is weak solu.

Prop: global Lip will lead to global solu, while local Lip may lead to local solu. Examples: $\dot{x}=1+x^2, x=\tan(t)$; $|f(t,x_1)-f(t,x_2)|\le A|x_1-x_2|^k$ with $k>1$.

For the former, consider a new norm $||x(t)||=\max_{t\in I}|x(t)e^{-\alpha t}|$. Def $Tx=x_0+\int_{t_0}^t f(t,x)dt$, $|Tx-Ty|\le L\int_{t_0}^t|(x-y)(s)|ds=\int_{t_0}^t|(x-y)(s)e^{-\alpha s}|e^{\alpha s}ds\le L||x-y||\int_{t_0}^t e^{\alpha s}ds\le\frac{L}{\alpha}e^{\alpha t}||x-y||\Rightarrow$

$||Tx - Ty|| \leq \frac{L}{\alpha}||x - y||$. Let $\alpha = 2L$ and get global Lip.

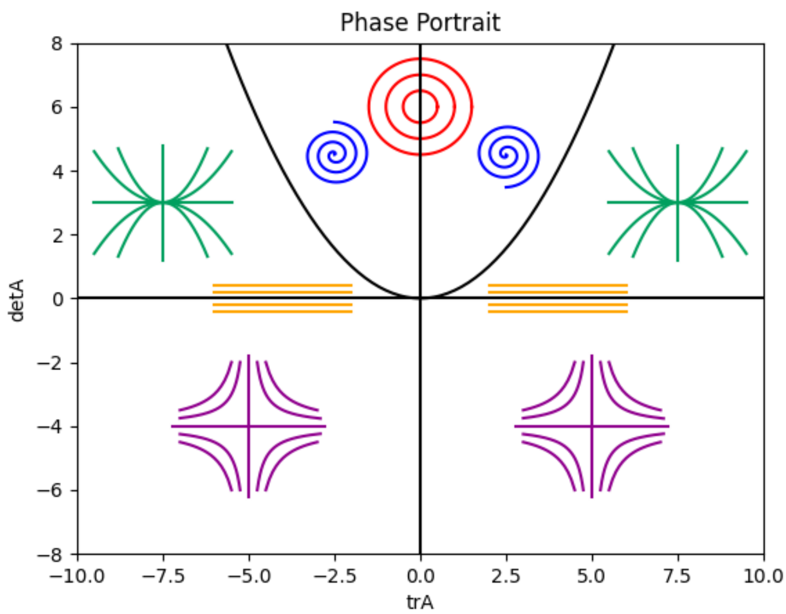
Consider $\begin{cases} \dot{x} = f(t, x) \\ x|_{t=0} = x_0 \end{cases}$ $\widetilde{x}_0 \to x_0$, we have $\widetilde{x} - x = \widetilde{x}_0 - x_0 + \int_0^t f(s, \widetilde{x}(s)) - f(s, x)ds$, $|\widetilde{x} - x| \leq |\widetilde{x}_0 - x_0| + L\int_0^t |\widetilde{x}(s) - x(s)|ds$, $|\widetilde{x} - x| \leq |\widetilde{x}_0 - x_0|e^{tL}$.

1.6 Qualitative Analysis

Examples: $\dot{x} = Ax$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$. Their phase portraits are parabolas, hyperbolas, circles and spirals.

$\dot{x} = f(x)$ where $f(x^*) = 0$. $\dot{\delta x} = f(x^* + \delta x) = (\delta x \cdot \nabla)f(x^*) + o(||\delta x||) \sim A\delta x$.

Consider dimension $= 2$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2}$. $\det(\lambda I - A) = \lambda^2 - \text{tr}(A)\lambda + \det(A), \Delta = (\text{tr}A)^2 - 4\det(A)$. See the following picture.



Def: L-Stable: $\forall \epsilon > 0, \exists \delta > 0, \text{s.t.} \forall x_0 \in O(x^*, \delta), |x(t) - x^*| < \epsilon$. A-Stable: $x(t) \to x^*$ with $t \to +\infty$.

Lyapunov 1st Thm: If $\text{Re}\lambda(A) < 0 \Rightarrow$ A-Stable.

Lyapunov 2nd Thm: $\text{Re}\lambda(A) < 0 \Leftrightarrow \exists G, G = G^*, G > 0$ s.t. $GA + A^*G < 0$.

Proof: $\Leftarrow$: $((GA + A^*G)h, h) = (GAh, h) + (A^*Gh, h) = \lambda(Gh, h) + (Gh, Ah) = (\lambda + \bar{\lambda})(Gh, h) = 2\text{Re}\lambda(Gh, h) < 0 \Rightarrow \text{Re}\lambda < 0$.

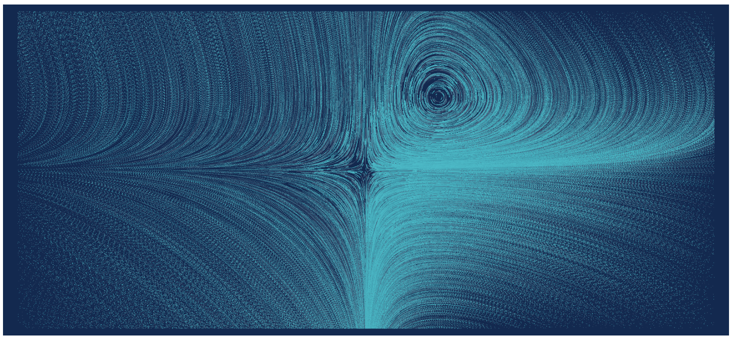
$\Rightarrow$: Let $G = \int_0^\infty e^{A^*t}e^{At}dt.(Gx, x) = \int_0^\infty ||e^{At}||^2 dt > 0, GA + A^*G = \int_0^\infty \frac{d}{dt}e^{A^*t}e^{At}dt = -I < 0$.
Remark: 1) G is not unique; 2) If $\text{Re}\lambda(A) < 0, \int_0^\infty e^{A^*t}e^{At}dt \sim \sum_n e^{A^*n}e^{An}$ is well defined.

Consider $\dot{x} = Ax + F(t, x)$. If $\lim_{x\to 0} \frac{|F(t,x)|}{|x|} = 0$, L-1st Thm is still valid. (Hint: Duhammel)

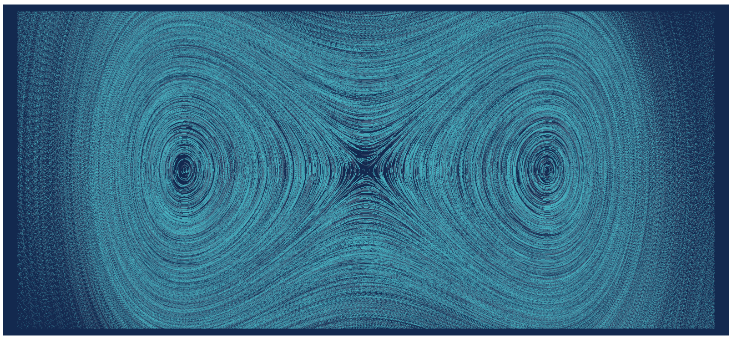
Example: $\begin{cases} \dot{x} = x(ay - b) \\ \dot{y} = y(c - dx) \end{cases}$, $a, b, c, d, > 0$. $\nabla F = \begin{pmatrix} ay - b & ax \\ -dy & c - dx \end{pmatrix}$.

$\nabla F|_{(0,0)} = \begin{pmatrix} -b & 0 \\ 0 & c \end{pmatrix} \to$ hyperbola. $\nabla F|_{(\frac{c}{a},\frac{b}{a})} = \begin{pmatrix} 0 & \frac{ac}{d} \\ -\frac{db}{a} & 0 \end{pmatrix} \to$ a center for linearized system.

In fact we have $\frac{dx}{dy} = \frac{x(ay-b)}{y(c-dx)}$, thus $V(x,y) = ay + dx - c\ln x - b\ln y$ where $\frac{d}{dt}V = 0, ay - b\ln y + dx - c\ln y = ay_0 - b\ln y_0 + dx_0 - c\ln x_0 \Rightarrow e^{d(x-x_0)}(\frac{x_0}{x})^c = e^{-a(y-y_0)}(\frac{y_0}{y})^b$.

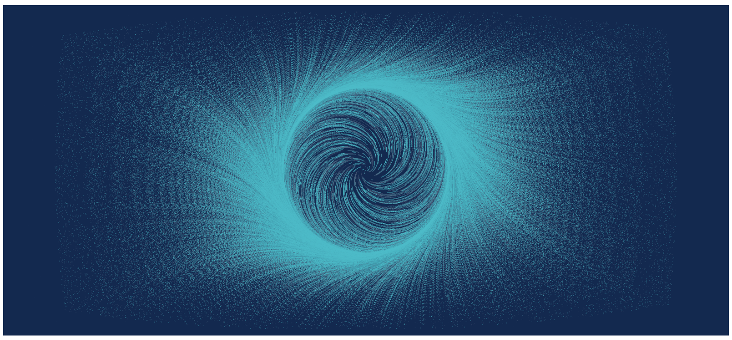


Example: $m\ddot{x} = -\frac{\partial V}{\partial x}, p = m\dot{x}, z = (x,p), H = \frac{p^2}{2m} + V(x)$. Let $V = \frac{1}{4}(1-x^2)^2, m = 1$, then $F = (p, x - x^3)^T, \nabla F = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}. \nabla F|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to$ saddle.$\nabla F|_{0,\pm 1} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$. In fact, $H = \frac{p^2}{2} + \frac{1}{4}(1-x^2)^2 \equiv C$.



Example (Bifurcation): $\frac{dx}{dt} = (a-1)x - x^3$. If $a < 1$, $\frac{df}{dx}|_{x=0} = a - 1 < 0 \to$ stable. If $a > 1$, $\frac{df}{dx}|_{x=0} > 0 \to$ unstable, $\frac{df}{dx}|_{x=\pm\sqrt{a-1}} = -2(a-1) < 0 \to$ stable. If $a = 1$, $\frac{dx}{dt} = -x^3 \to x = 0$ is stable.

Example: $\begin{cases} \dot{x} = ax - y - x(x^2+y^2) \\ \dot{y} = x + ay - y(x^2+y^2) \end{cases}$. Let $x = r(t)\cos\theta(t), y = r(t)\sin\theta(t)$, then $\begin{cases} \dot{r} = ar - r^3 \\ \dot{\theta} = 1 \end{cases}$, $r^* = \sqrt{a} \to$ limiting circle.



Lyapunov function method: $L(x,y) = L(x(t),y(t))$. 1) L is $C^1, L(x^*,y^*) = 0, L(x,y) \neq 0$ if $(x,y) \neq (x^*,y^*)$. 2) $\frac{d}{dt}L \begin{cases} < 0 \to \text{ A-stable} \\ \leq 0 \to \text{ stable} \\ > 0 \to \text{ unstable} \end{cases}$. $\frac{d}{dt}L = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = f\frac{\partial L}{\partial x} + g\frac{\partial L}{\partial y} = (F\cdot\nabla)L, F = (f,g)$. $L^{-1}(c) \to$ closed curve (for enough small c).

Example: $\dot{x} = -\nabla V(x). L = V - V(x^*), \dot{L} = \dot{V} = \nabla V \cdot \dot{x} = -|\nabla V|^2 \leq 0$.

Example: $\begin{cases} \dot{x} = xf(y) \\ \dot{y} = yg(x) \end{cases} \Rightarrow \frac{dx}{dy} = \frac{xf(y)}{yg(x)} \Rightarrow \frac{f(y)}{y}dy = \frac{g(x)}{x}dx$. Let $V = \int_0^y \frac{f(t)}{t}dt - \int_0^x \frac{g(s)}{s}ds, L = V - V(x^*,y^*) \Rightarrow \dot{L} = \dot{V} = 0$.

Example: $\begin{cases} \dot{x} = -2xy \\ \dot{y} = x^2 - 2y^3 \end{cases}$, $(x^*,y^*) = (0,0), \nabla F|_{(0,0)} = 0$. We guess $L = ax^{2m} + by^{2n}, \dot{L} = 2max^{2m-1}\dot{x} + 2nby^{2n-1}\dot{y} = -4max^{2m}y + 2nbx^2y^{2n-1} - 4nby^{2n+2}$. Let $4am = 2nb, 2m = 2, 2n-1 = 1 \Rightarrow m = n = 1, b = 2a$, thus $L = x^2 + 2y^2, \dot{L} = -8y^4 \leq 0$.

Example: $\ddot{x} = -\nabla V, V \in C^2$. If $(x\cdot\nabla)V \leq -CV (C \geq 2)$, then $\partial_t^2(|x(t)|^2) = 2\partial_t(x,\dot{x}) = 2|\dot{x}|^2 + 2x\ddot{x} = 2|\dot{x}|^2 - 2(x\cdot\nabla)V(x) \geq 2|\dot{x}|^2 + 2CV = 4(\frac{1}{2}|\dot{x}|^2 + \frac{C}{2}V) \geq 4E(t) \geq 0 \Rightarrow |x(t)|^2$ is convex.

2 Asymptotic Analysis

2.1 Regular Expansion

$\mathscr{L}_\epsilon = \mathscr{L}_0 + \epsilon\mathscr{L}_1$ where $\mathscr{L}_0, \mathscr{L}_1$ is differential operator.

ANSATZ: $y_\epsilon = y_0 + \epsilon y_1 + \cdots \Rightarrow (\mathscr{L}_0 + \epsilon\mathscr{L}_1)(y_0 + \epsilon y_1 + \cdots) = f$. $O(1) : \mathscr{L}_0 y_0 = f$; $O(\epsilon) : \mathscr{L}_0 y_1 + \mathscr{L}_1 y_0 = 0$; $O(\epsilon^2) : \mathscr{L}_0 y_2 + \mathscr{L}_1 y_1 = 0; \cdots$. Assume $\mathscr{L}_0^{-1}\exists$, so $y_0 = \mathscr{L}_0^{-1}f, y_1 = -\mathscr{L}_0^{-1}\mathscr{L}_1 y_0 = -\mathscr{L}_0^{-1}\mathscr{L}_1\mathscr{L}_0^{-1}f$.

Example: $\begin{cases} \ddot{x} + \sin x = 0 \\ x(0) = A, \dot{x}(0) = 0 \end{cases}$ where A is small. Question: $T(A) = ?$

Assume $x = x_0 + Ax_1 + A^2x_2 + A^3x_3 + o(A^4), \sin x = \sin(x_0 + Ax_1 + A^2x_2 + A^3x_3 + o(A^4)) = \sin x_0 + \cos x_0(Ax_1 + A^2x_2 + A^3x_3 + o(A^4)) + \frac{1}{2}(-\sin x_0)(Ax_1 + A^2x_2 + A^3x_3 + o(A^4))^2 + \frac{1}{6}(-\cos x_0)(Ax_1 + A^2x_2 + A^3x_3 + o(A^4))^3 + o(A^4)$. $O(1) : \begin{cases} \ddot{x}_0 + \sin x_0 = 0 \\ x_0(0) = 0, \dot{x}_0(0) = 0 \end{cases} \Rightarrow x_0 \equiv 0$; $O(A) : \begin{cases} \ddot{x}_1 + x_1 = 0 \\ x_1(0) = 1, \dot{x}_1(0) = 0 \end{cases} \Rightarrow x_1(t) = \cos t$; $O(A^2) : \begin{cases} \ddot{x}_2 + x_2 = 0 \\ x_2(0) = 0, \dot{x}_2(0) = 0 \end{cases} \Rightarrow x_2 \equiv 0$; $O(A^3) : \begin{cases} \ddot{x}_3 + x_3 - \frac{1}{6}x_1^3 = 0 \\ x_3(0) = 0, \dot{x}_3(0) = 0 \end{cases} \Rightarrow x_3(t) = \frac{1}{6}\int_0^t \sin(t-s)\cos^3 s ds = -\frac{1}{192}(\cos 3t - \cos t) + \frac{1}{16}t\sin t$. Thus $x(t) = A\cos t + A^3(-\frac{1}{192}(\cos 3t - \cos t) + \frac{1}{16}t\sin t) + o(A^4)$. $\dot{x}(T(A)) = 0, T(A) = 2\pi + B, B << 1, \sin B \sim B, \cos B \sim 1$. By specific computing, we have $T(A) = 2\pi(1 + \frac{1}{16}A^2 + o(A^4))$.

2.2 Singular Perturbation

We will begin with a special example and introduce multiple-scale expansion in sequence.

Example: $\begin{cases} \ddot{x}+(1+\epsilon)x=0 \\ x(0)=1, \dot{x}(0)=0 \end{cases}$. $x_\epsilon(t)=\cos(\sqrt{1+\epsilon}t)$. $\epsilon\to 0, x_0(t)=\cos t, x_\epsilon(t)\to x_0(t)$. We have estimation $|x_\epsilon(t)-x_0(t)|\le Ct\epsilon$. $x_\epsilon(t)=x_0+\epsilon x_1+\epsilon^2 x_2+\cdots$, then $O(1):\ddot{x}_0+x_0=0$; $O(\epsilon)$: $\ddot{x}_1+x_1=-x_0;\cdots$. $x_\epsilon(0)=1,\dot{x}_\epsilon(0)=0\Rightarrow x_0(0)=1, x_i(0)=0(i\ge 1),\dot{x}_j(0)=0(j\ge 0)$. $x_0(t)=\cos t, x_1(t)=-\frac{t}{2}\sin t\Rightarrow x_\epsilon(t)\sim\cos t-\frac{1}{2}\epsilon t\sin t+\cdots$. We notice that if $t\sim O(\frac{1}{\epsilon})$, the conventional method does not work.

Denote $\tau=\epsilon t, x(t)\to Z(t,\tau), \frac{d}{dt}=\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial\tau}, \frac{d^2}{dt^2}=\frac{\partial^2}{\partial t^2}+2\epsilon\frac{\partial^2}{\partial t\partial\tau}+\epsilon^2\frac{\partial^2}{\partial\tau^2}$. The PLK method has two steps: 1) Operator expansion; 2) $Z(t,\tau)=Z_0(t,\tau)+\epsilon Z_1(t,\tau)+O(\epsilon^2)$. When computing, we must remove the singular term (e.g. $e^{i\omega t}$) which may make the solution drift.

$(\frac{\partial^2}{\partial t^2}+(1+\epsilon))Z=(\frac{\partial^2 Z}{\partial t^2}+Z)+\epsilon(2\frac{\partial^2 Z}{\partial t\partial\tau}+Z)+\epsilon^2\frac{\partial^2 Z}{\partial\tau^2}=0$. $O(1)$: $\frac{\partial^2 Z_0}{\partial t^2}+Z_0=0$; $O(\epsilon)$: $\frac{\partial^2 Z_1}{\partial t^2}+Z_1=-Z_0-2\frac{\partial^2 Z_0}{\partial t\partial\tau}$. $Z_0=A(\tau)e^{it}+A^*(\tau)e^{-it}$, $\frac{\partial^2 Z_1}{\partial t^2}+Z_1=[-A(\tau)-2iA'(\tau)]e^{it}+[-A^*(\tau)+2i(A^*(\tau))']e^{-it}$. We hope $A(\tau)+2iA'(\tau)\equiv 0\Rightarrow A'(\tau)=-\frac{1}{2i}A(\tau)\Rightarrow A(\tau)=A(0)e^{\frac{1}{2}i\tau}$. Because IC $Z_0(0,0)=1, \frac{\partial Z_0}{\partial t}(0,0)=0\Rightarrow Z_0(t,\tau)=\cos(1+\frac{\epsilon}{2})t$. We have the estimation $|x_\epsilon-Z_0-\epsilon Z_1|\le c(t)\epsilon^2$ where $t\sim O(\frac{1}{\epsilon})$.

Example: $\begin{cases} \ddot{x}+x+\epsilon x^3=0 \\ x(0)=1, \dot{x}(0)=0 \end{cases}$. $\tau=\epsilon t$, $\begin{cases} O(1): \frac{d^2 Z_0}{dt^2}+Z_0=0 \\ O(\epsilon): \frac{d^2 Z_1}{dt^2}+Z_1=-2\frac{\partial^2 Z}{\partial t\partial\tau}-Z_0^3 \end{cases}$. $Z_0=A(\tau)e^{it}+A^*(\tau)e^{-it}$, $-2\frac{\partial^2 Z_0}{\partial t\partial\tau}-Z_0^3=[-3A^2(\tau)A^*(\tau)-2iA'(\tau)]e^{it}+[-3A(\tau)(A^*(\tau))^2+2i(A^*(\tau))']e^{-it}-A^3(\tau)e^{3it}-(A^*(\tau))^3e^{-3it}\Rightarrow A'(\tau)=\frac{3}{2}iA^2(\tau)A^*(\tau)$. Let $A(\tau)=R(\tau)e^{i\theta(\tau)}\Rightarrow\dot{R}(\tau)+iR(\tau)\dot{\theta}(\tau)=\frac{3}{2}iR^3(\tau)\Rightarrow\dot{R}(\tau)=0,\dot{\theta}(\tau)=\frac{3}{2}R^2(\tau)\Rightarrow R(\tau)=R(0),\theta(\tau)=\theta(0)+\frac{3}{2}R^2(0)\tau, A(\tau)=R(0)e^{i(\theta(0)+\frac{3}{2}R^2(0)\tau)}, Z_0=R(0)e^{i(t+\theta(0)+\frac{3}{2}R^2(0)\tau)}+R(0)e^{-i(t+\theta(0)+\frac{3}{2}R^2(0)\tau)}$. Beacause IC $Z_0(0,0)=0, \frac{\partial Z_0}{\partial t}(0,0)=0\Rightarrow R(0)=\frac{1}{2},\theta(0)=0\Rightarrow Z_0(t,\tau)=\cos(1+\frac{3}{8}\epsilon)t$.

Remark: PLK method only works for $t\sim O(\epsilon^{-k})$.

3 Derivation of Partial Differential Equation

3.1 Conservation Law

Consider traffic flow(flux). $\rho(x,t)\to$ density of cars, $J(\rho)\to$ flow function. So $\frac{d}{dt}\int_a^b\rho(x,t)dx=J(a)-J(b)=J(\rho(a,t))-J(\rho(b,t))=-\int_a^b J'(\rho)dx\Rightarrow\frac{d}{dt}\int_a^b\partial_t\rho+\partial_x J=0\Rightarrow\partial_t\rho+\partial_x J=0$. $J(\rho)=C\rho\Rightarrow\partial_t\rho+\partial_x(C\rho)=0$. $J(\rho)=C\rho(M-\rho)$ and $\bar{\rho}=\rho/M\Rightarrow\partial_t\bar{\rho}+\partial_x[\beta\bar{\rho}(1-\bar{\rho})]=0,\beta=CM$. $J(\rho)=C\rho-\mu\rho_x\Rightarrow\partial_t\rho+\partial_x(C\rho)-\mu\rho_{xx}=0$. $J(\rho)=\frac{\rho^2}{2}-\mu\rho_x\Rightarrow\partial_t\rho+\frac{1}{2}(\rho^2)_x-\mu\rho_{xx}=0$.

For high-dimension cases, $\frac{d}{dt}\int_\Omega\rho(x,t)dx=-\int_{\partial\Omega}J\cdot\vec{n}dx=-\int_\Omega\nabla\cdot Jdx\Rightarrow\partial_t\rho+\nabla\cdot J=0$.

Example: String Vibration: 1) Homogeneous: $\rho(x)\equiv\rho$; 2)Elastic String (only stretching, no shearing, no bending). $u(x,t)=$ vertical distance at (x,t), $|\partial_x u|<<1$. $T(x)=$ tension, $|\partial_x u|<<1$ $\Rightarrow T(x)\approx T(x+\Delta x)\Rightarrow T(x)\equiv T$. So $\int_x^{x+\Delta x}\rho\frac{\partial^2 u}{\partial t^2}dx=T\sin\alpha(x+\Delta x)-T\sin\alpha(x)=T\partial_x u(x+\Delta x,t)-T\partial_x u(x,t)=T\int_x^{x+\Delta x}\frac{\partial^2 u}{\partial x^2}dx\Rightarrow\rho\frac{\partial^2 u}{\partial t^2}=T\frac{\partial^2 u}{\partial x^2}$. If $E=$ Young's modulus, then $\rho S(x)\frac{\partial^2 u}{\partial t^2}=E\frac{\partial}{\partial x}(S(x)\frac{\partial u}{\partial x})$.

For high-dimension cases, $\int_x^{x+\Delta x}\int_y^{y+\Delta y}\rho\frac{\partial^2 u}{\partial t^2}dydx=T(\frac{\partial u}{\partial x}(x+\Delta x,y,t)-\frac{\partial u}{\partial x}(x,y,t))\Delta y+T(\frac{\partial u}{\partial y}(x,y+\Delta y,t)-\frac{\partial u}{\partial y}(x,y,t))\Delta x\Rightarrow\frac{1}{c^2}\partial_t^2 u=\triangle u$ where $c^2=\frac{T}{\rho}$. For this class, we will impose some BC/IC.

BC: 1) Dirichlet BC: $u(0,t)=u(L,t)=0$ (homo) or $u(0,t)=f(t),u(L,t)=g(t)$ (inhomo).

2) Neumann BC: $u_x'(0,t) = u_x'(L,t) = 0$ or $-Tu_x'(0,t) = f(t), -Tu_x'(L,t) = g(t)$.

3) Robin BC: $\alpha u(0,t) + \beta u_x(0,t) = \alpha u(L,t) + \beta u_x(L,t) = 0$.

4) Newton's Law of Cooling: $-\frac{k\alpha}{\beta}(u(0,t) - g(t)) = -ku_x(0,t)$.

3.2 Calculus of Variation

Consider energy function $I[u] = \frac{1}{2}\int_0^1 |u'|^2 dx - \int_0^1 f(x)u(x)dx$, our goal is to $\min_{u\in A} I[u]$ where A is admissible set/space. We asuume the minimizer u exists. $u_t = u + tv, I[u_t] \geq I[u]$. We denote $g(t) = I[u_t]$, then $g(t) \geq g(0), g'(0) = 0$. $g(t) = \frac{1}{2}\int_0^1 |u' + tv'|^2 dx - \int_0^1 f(u+tv)dx, g'(t) = \int_0^1 u'(x)v'(x)dx + t\int_0^1 (v')^2 dx - \int_0^1 f(x)v(x)dx, g'(0) = 0 \Rightarrow \int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx$ (Euler-Lagrange EQ) $\rightarrow$ weak solu. $\int_0^1 u'(x)v'(x)dx = \int_0^1 u'(x)dv(x) = u'(1)v(1) - u'(0)v(0) - \int_0^1 u''(x)v(x)dx$ $= \int_0^1 f(x)v(x)dx \Rightarrow u'(1)v(1) - u'(0)v(0) = \int_0^1 (u'' + f)vdx$.

1) Assume $A = \{u \in C^2(0,1) | I[u] < \infty, u'(0) = u'(1) = 0\}$, we have $\begin{cases} -u'' = f \\ u'(0) = u'(1) = 0 \end{cases}$ $\rightarrow$ Neumann BVP.

2) Assume $A = \{u \in C^2(0,1) | I[u] < \infty, u(0) = A, u(1) = B\}$. In particular, $u, u_t \in A$, so we have $v(0) = v(1) = 0 \Rightarrow \begin{cases} -u'' = f \\ u(0) = A, u(1) = B \end{cases}$ $\rightarrow$ Dirichlet BVP.

3) Assume $A = \{u \in C^2(0,1) | I[u] < \infty, u'(0) + a(0)u(0) = u'(1) + a(1)u(1) = 0\}$ and consider $I[u] = \frac{1}{2}\int_0^1 |u'(x)|^2 dx - \int_0^1 f(x)u(x)dx + \frac{1}{2}[a(1)u^2(1) - a(0)u^2(0)]dx$. By computing $g(t) = I[u + tv], g'(0) = 0$, we have $\begin{cases} -u'' = f \\ u'(0) + a(0)u(0) = u'(1) + a(1)u(1) = 0 \end{cases}$ $\rightarrow$ Robin BC.

Example: $I[u] = \frac{1}{2}\int_\Omega \mathcal{C}\epsilon(u):\epsilon(u)dx - \int_\Omega f(x)u(x)dx$. $\mathcal{C}A = 2\mu A + \lambda \mathrm{tr}(A)I, A \in R^{2\times 2}, \mu, \lambda =$ Lamé constants > 0,$u = (u_1, u_2) =$ displacement,$fu = f_1u_1 + f_2u_2, \epsilon(u) =$ strain tensor $\in R^{2\times 2} = \frac{1}{2}(\nabla u + \nabla u^T)$. Denote $g(t) = \frac{1}{2}\int_\Omega \mathcal{C}\epsilon(u+tv):\epsilon(u+tv)dx - \int_\Omega f(x)(u(x) + tv(x))dx, \epsilon(u+tv) = \epsilon(u) + t\epsilon(v)$, then $g'(0) = \frac{1}{2}\int_\Omega \mathcal{C}\epsilon(u):\epsilon(v)dx + \frac{1}{2}\mathcal{C}\epsilon(v):\epsilon(u)dx - \int_\Omega f(x)v(x)dx = 0 \Rightarrow \int_\Omega \mathcal{C}\epsilon(u):\epsilon(v)dx = \int_\Omega f(x)v(x)dx$. Define $\sigma := \mathcal{C}\epsilon(u) =$ stress tensor, then $\sigma = \sigma^T, \int_\Omega \sigma:\epsilon(v)dx = \frac{1}{2}\int \sigma:(\nabla v + \nabla v^T)dx = \int \sigma:\nabla v dx = -\int_\Omega \nabla \cdot \sigma v dx + \int_{\partial\Omega}(\sigma \cdot n)v d\sigma(x) = \int_\Omega f(x)v(x) = 0 \Rightarrow -\nabla\cdot\sigma = f, \sigma \cdot n = 0$ (Navier EQ). Here, $A:B = \mathrm{tr}(AB^T)$.

Least action pinciple: $A[u] = \int_{t_0}^{t_1}(K(t) - P(t))dt$. $A \rightarrow$ action, $K \rightarrow$ kinectic, $P \rightarrow$ potential. Motion path u always follows the extreme point of $A[u]$.

Example: $\min_\theta \int_{t_0}^{t_1}(\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta))dt$. By computing we get $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

Consider $L = \int_{t_0}^{t_1} \mathscr{L}(t, q, \dot{q})dt$ where $\mathscr{L} \rightarrow$ Lagrangian. By computing we get $\frac{\partial \mathscr{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}}$ (E-L EQ). $p = \frac{\partial \mathscr{L}}{\partial \dot{q}} \rightarrow$ Generalized Momentum, $\mathscr{H} = p \cdot \dot{q} - \mathscr{L} \rightarrow$ Hamiltonian. Using E-L EQ, we have $\frac{\partial \mathscr{H}}{\partial p_i} = \dot{q}_i, \frac{\partial \mathscr{H}}{\partial q_i} = -\dot{p}_i \Rightarrow \frac{d\mathscr{H}}{dt} = \dot{p}_i \cdot \dot{q}_i - \frac{\partial \mathscr{L}}{\partial p_i}\dot{p}_i = 0$.

Example: $\begin{cases} \dot{x}(s) = f(x(s), \alpha(s)), t < s < T \\ x(t) = x \end{cases}$ where $\alpha \rightarrow$ control and $x \rightarrow$ response. Consider $u(x,t) = \inf_\alpha \int_t^T W(x(s), \alpha(s))ds + g(x(T))$ where $W \rightarrow$ running cost and $g \rightarrow$ final cost. Admissible solution set $A = \{\alpha | \alpha \text{ is measurable}\}$. $\forall h > 0, u(x,t) = \inf \int_t^{t+h} W(x(s), \alpha(s))ds + u(x(t+h), t+h)$. By Taylor expansion, we have $x(t+h) \approx x(t) + \dot{x}(t)h = x(t) + f(x(t), \alpha(t))h, u(x(t+h), t+h) \approx$

$u(x(t)+f(x,\alpha)h,t+h) \approx u(x(t),t)+\partial_x uf(x,\alpha)h+\partial_t uh$. Then $u(x,t) \approx \inf_\alpha \int_t^{t+h} W(x(s),\alpha(s))ds + u(x,t) + \partial_x uf(x,\alpha)h + \partial_t uh \Rightarrow \partial_t u + \inf_\alpha(\partial_x uf + W(x,\alpha)) = 0$. Define $H(p) = \inf_\alpha(pf(x,\alpha) + W(x,\alpha))$, we get $\partial_t u + H(\partial_x u) = 0 \rightarrow$ Hamilton-Jacobi EQ. If $f(x,\alpha) = -\alpha, W(x,\alpha) = \frac{1}{2}\alpha^2$, we have $\partial_t u - \frac{1}{2}(\partial_x u)^2 = 0$.

Consider $A(a,\alpha), B(b,\beta)$, we want to find the shortest route connecting A with B. $I[u] = \int ds = \int_a^b \sqrt{1+u'(x)^2}dx$, admissible solution set $A = \{u \in C^1 | u(a) = \alpha, u(b) = \beta\}$. Denote $v_t = u + tv$ where u is minimizer and v is test function, $g(t) = \int_a^b \sqrt{1+(u'+tv')^2}dx$, $g'(0) = 0 \Rightarrow \int_a^b \frac{u'v'}{\sqrt{1+(u')^2}}dx = 0 = -\int_a^b \frac{d}{dx}(\frac{u'}{\sqrt{1+u^2}})vdx \Rightarrow -\frac{d}{dx}(\frac{u'}{\sqrt{1+(u')^2}}) = 0 \Rightarrow u'' = 0$.

Consider ray-propagation rules. $T = \int dt = \int \frac{ds}{c(x,y)} = \int_a^b \frac{\sqrt{1+u'^2}}{c(x,u(x))}dx = \int_a^b n(x,u(x))\sqrt{1+u'^2}dx$. $g(t) = \int_a^b n(x,u+tv)\sqrt{1+(u'+tv')^2}dx, g'(t) = \int_a^b \frac{\partial n}{\partial y}\sqrt{1+v_t^2}v + n\frac{(u'+tv')v'}{\sqrt{1+(u'+tv')^2}}dx$. $g'(0) = 0 \Rightarrow \int_a^b \frac{\partial n}{\partial y}(x,u)\sqrt{1+u'^2}v + n\frac{u'v'}{\sqrt{1+u'^2}}dx \Rightarrow -\frac{d}{dx}(\frac{u'}{\sqrt{1+u'^2}}) + \partial_y n\sqrt{1+u'^2} = 0$.

A special case is $n = \begin{cases} n_1, a < x < x_0 \\ n_2, x_0 < x < b \end{cases}$. Then $0 = g'(0) = \int_a^b \frac{nu'v'}{\sqrt{1+u'^2}}dx = \int_a^{x_0} \frac{n_1u'v'}{\sqrt{1+u'^2}}dx + \int_{x_0}^b \frac{n_2u'v'}{\sqrt{1+u'^2}}dx \Rightarrow \frac{n_1u'}{\sqrt{1+u'^2}}(x_0^-) - \frac{n_2u'}{\sqrt{1+u'^2}}(x_0^+) = 0$. Explanation: $\frac{n_1u'}{\sqrt{1+u'^2}} = \frac{n_1u'dx}{\sqrt{1+u'^2}dx} = n_1\frac{dy}{ds} = n_1\sin\theta_1 = n_2\sin\theta_2$, i.e. refraction law.

As for high-dimension cases, $I[u] = \int_\Omega \sqrt{1+|\nabla u|^2}dx$, E-L equation: $\int_\Omega \frac{\nabla u\cdot\nabla v}{\sqrt{1+|\nabla u|^2}}dx = 0 \Rightarrow -\nabla\cdot(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0 (|\nabla u| << 1) \Rightarrow \triangle u = 0$.

Consider $\min_{u\in A} I[u] = \int_a^b \sqrt{1+u'^2}dx$ s.t.$\int_a^b u(x)dx = A$. $I_\lambda[u] = I[u] + \lambda(\int_a^b u(x)dx - A), g'(0) = 0 \Rightarrow \int_a^b \frac{u'v'}{\sqrt{1+u'^2}} + \lambda udx = 0 \Rightarrow -\frac{d}{dx}(\frac{u'}{\sqrt{1+u'^2}}) + \lambda = 0 \Rightarrow \frac{u'}{\sqrt{1+u'^2}} = \lambda x + \mu \Rightarrow \frac{u'^2}{1+u'^2} = (\lambda x + \mu)^2 \Rightarrow u' = \frac{\lambda x+\mu}{\sqrt{1-(\lambda x+\mu)^2}} \Rightarrow u(x) - \alpha = \int_a^x \frac{\lambda y+\mu}{\sqrt{1-(\lambda y+u)^2}}dy = (\lambda^{-2} - (x+\frac{\mu}{\lambda}^2))^{\frac{1}{2}} - (\lambda^{-2} - (a+\frac{\mu}{\lambda}^2))^{\frac{1}{2}} \Rightarrow (u(x) - (\alpha - \alpha_0)^2) + (x+\frac{\mu}{\lambda})^2 = \lambda^{-2} \rightarrow$ circle.

Consider $\min_{u\in A} Q[u] = \frac{\int|\nabla u|^2dx}{\int u^2dx}$ where $A = H_0^1(\Omega) = \{u \in L^2, \nabla u \in L^2\}$. $Q[u+tv] = \frac{\int_\Omega |\nabla(u+tv)|^2dx}{\int_\Omega |u+tv|^2dx} = \frac{\int_\Omega |\nabla u|^2dx + 2t\int_\Omega \nabla u\cdot\nabla vdx + t^2\int_\Omega |\nabla v|^2dx}{\int u^2dx + 2t\int_\Omega uvdx + t^2\int_\Omega v^2dx}$. $g'(0) = 0 \Rightarrow \int_\Omega \nabla u\cdot\nabla vdx = Q[u]\int_\Omega u\cdot vdx$. Denote $Q[u] = \lambda$, we get $-\triangle u = \lambda u, x \in \Omega, u = 0, x \in \partial\Omega$. If $\Omega = (0,\pi)$, then $\min\lambda = 1, Q[u] \geq 1, \int_0^\pi u^2dx \leq \int_0^\pi (u')^2dx$ (Poincare inequality). If $\Omega = (0,L)$, then $\lambda_k = (\frac{k\pi}{L})^2, \int_0^L u^2dx \leq (\frac{L}{\pi})^2\int_0^L u'^2dx$.

3.3 Second-order Variation

For general case, $I[u] = \int_a^b L(u,u')dx$. $g(t) = \int_a^b L(u+tv, u'+tv')dx$. Let $p = u'$, then $g'(t) = \int_a^b \frac{\partial L}{\partial u}v + \frac{\partial L}{\partial p}vdx, g''(t) = \int_a^b (\frac{\partial^2 L}{\partial u^2}v^2 + 2\frac{\partial^2 L}{\partial u\partial p}vv' + \frac{\partial^2 L}{\partial p^2}v'^2)dx$. $g'(0) = \int_a^b Av^2 + 2Bvv' + Cv'^2dx$ where $A = \frac{\partial^2 L}{\partial u^2}(u,u'), B = \frac{\partial^2 L}{\partial u\partial p}(u,u'), C = \frac{\partial^2 L}{\partial p^2}(u,u')$. It is difficult to strictly prove $g''(0) \geq 0$, but when $L = (1+p^2)^{\frac{1}{2}}$, $A = B = 0, C = \frac{1}{(1+p^2)^{\frac{3}{2}}} > 0$, thus $g''(0) \geq 0$.

Thm: Let $Q[u] = \int_a^b Au^2 + 2Buu' + C(u')^2dx$ where $A, B, C \in C^1(a,b)$, then $Q[u] \geq 0$ for all u satisfying $u(a) = u(b) = 0$ if 1) $C \geq 0$ (Legendre condition); 2) $\forall c \in (a,b)$, the equation $-(Cz')' + (A-B')z = 0, z(a) = z(c) = 0$ only has a trivial solution, i.e. there exists no conjugate point $c \in (a,b)$.

Proof: $g''(t) = \int_a^b Au^2 + B(u^2)' + C(u')^2dx = \int_a^b C(u')^2 + (A-B')u^2dx = \int_a^b pu'^2 + qu^2dx (u(a) = u(b) = 0) = \int_a^b (pu'^2 + qu^2)dx + \int_a^b (wu^2)'dx = \int_a^b pu'^2 + qu^2 + w'u^2 + 2wuu'dx = \int_a^b p(u'^2 + \frac{2w}{p}uu' +$

$\frac{w'+q}{p}u')dx = \int_a^b p(u' + \frac{w}{p}u)^2 dx \Rightarrow \frac{w'+q}{p} = (\frac{w}{p})^2 \Leftrightarrow p(q+w) = w^2$ (Riccati EQ). Let $w = -\frac{z'}{z}p, w' = -\frac{z''z-(z')^2}{z^2}p - \frac{z'}{z}p'$, then $pq + pw' = pq - (\frac{p}{q})^2 + (\frac{pz'}{z})^2 - \frac{z'}{z}p'p = (\frac{z'}{z}p)^2 \Rightarrow -pz'' - p'z' + qz = 0 \Rightarrow -\frac{d}{dx}(pz') + qz = 0$. This equation with BC $z(a) = z(c) = 0$ has only zero equation, thus for all the BCs, it has a unique solution. In this way we can find a proper z and w and get the above-mentioned theorem.

Example: $Q[u] = \int_0^1 u'^2 - u^2 dx, u(0) = u(1) = 0$. $-u'' - u = 0, u(0) = u(1) = 0$ only has zero solution, so $Q[u] \geq 0$. But when modifying 1 with 4, there exists a conjugate point $\pi \in (0, 4)$.

3.4 Well-posedness of PDEs

$$\begin{cases} \text{1) existence: classical, weak, strong; domain}\Omega\text{, bound}\partial\Omega \\ \text{2) uniqueness: } \begin{cases} -\triangle u + u = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases} \text{ or } \begin{cases} -\nabla \cdot (A(x)\nabla u) = f \text{ in } \Omega \\ \partial_n u = g \text{ on } \partial\Omega \end{cases} A(x) \begin{cases} \text{bounded} \\ \text{measurable} \end{cases} \\ \text{3) smoothness: how solution changes when } t \text{ increases or decreases} \\ \text{4) stability: how solution changes when data is given a small pertubation} \end{cases}$$

Example: $AX = b$. The solution $A^{-1}b$ might be singular when $A = \text{diag}(1, \cdots, 1, \epsilon)$.

Example: $\triangle u = 0, u(x, 0) = 0$ and solutions are $u_n(x, y) = \frac{1}{n}e^{-n}\sin(nx)\sinh(ny)$.

4 Method of Characteristic

4.1 Simple First Order PDE

A PDE strained in the characteristic line might perform a behavior which can be described as an ODE. So we can transform PDE to ODE.

Consider conversation law (1st PDE): $\partial_t \rho + \partial_x f(\rho) = 0$. A special case is $f(\rho) = \rho$ and $\rho|_{t=0} = \rho_0(x)$. Define $x = Z(t)$ and $p(t) = \rho(Z(t), t)$, then $\dot{p}(t) = \partial_x \rho \dot{Z} + \partial_t \rho = \partial_x \rho \dot{Z} - \partial_x \rho = \partial_x \rho(\dot{Z} - 1)$. Characteristic line is $\dot{Z} = 1, \dot{p}(t) = 0 \Rightarrow Z(t) = Z(0) + t, p(t) = p(0) = \rho(Z(0), 0) = \rho_0(Z(0)) = \rho_0(Z(t) - t) \Rightarrow \rho(x, t) = \rho_0(x - t)$.

Example: $\partial_t \rho - \partial_x \rho = 0, \rho|_{t=0} = \rho_0(x)$. $p(t) = \rho(Z(t), t), \dot{p} = \partial_x \rho \dot{Z} + \partial_t \rho = \partial_x \rho(\dot{Z} + 1)$, The characteristic line is $\dot{Z} = -1, \dot{p}(t) = 0 \Rightarrow Z(t) = Z(0) - t, p(t) = p(0) = \rho(Z(0), 0) = \rho_0(Z(t) + t) \Rightarrow \rho(x, t) = \rho_0(x + t)$.

Example: $\partial_t \rho + a(x)\partial_x \rho = 0$ and $a(x) = x$. $\dot{p}(t) = \partial_x \rho \dot{Z} + \partial_t \rho = \partial_x \rho(\dot{Z} - a(Z)) = \partial_x \rho(\dot{Z} - Z)$. The characteristic line is $\dot{Z} = Z, \dot{p} = 0 \Rightarrow Z(t) = Z(0)e^t, p(t) = p(0) = \rho_0(Z(0)) = \rho_0(Z(t)e^{-t}) \Rightarrow \rho(x, t) = \rho_0(xe^{-t})$.

Example: $\partial_t u = A\partial_x u$ where $A \in R^{2\times 2}$ is constant matrix and $P^{-1}AP = \Lambda = \text{diag}(\lambda_1, \lambda_2)$. Then $\partial_t u = P\Lambda P^{-1}\partial_x u, P^{-1}\partial_t u = \Lambda P^{-1}\partial_x u \Rightarrow \partial_t W = \Lambda \partial_x W$ where $W = P^{-1}u$. The following process is familiar.

Another case is $\partial_t u_1 = \lambda \partial_x u_1 + \partial_x u_2, \partial_t u_2 = \lambda \partial_x u_2$ (i.e. A can not be diagonalized). $u_2(x, t) = \rho_2^2(x + \lambda t)$, thus $\partial_t u_1 = \lambda \partial_x u_1 + (\rho_2^2(x + \lambda t))'$. Consider the common case: $\partial_t \rho + \partial_x \rho = f(x, t)$, we have $\dot{p}(t) = \partial_x \rho \dot{Z} + \partial_t \rho + \partial_x \rho \dot{Z} + f(Z(t), t) - \partial_x \rho = \partial_x \rho(\dot{Z} - 1) + f(Z(t), t)$. Then $\dot{Z} = 1, \dot{p}(t) =$

$f(Z(t),t) \Rightarrow Z(t) = Z(0)+t, p(t) = p(0)+\int_0^t f(Z(\tau),\tau)d\tau = \rho_0(Z(0))+\int_0^t f(Z(\tau),\tau)d\tau = \rho_0(Z(t)-t)+\int_0^t f(Z(\tau),\tau)d\tau = \rho_0(Z(t)-t)+\int_0^t f(Z(0)+\tau,\tau)d\tau$. So $\rho(x,t) = \rho_0(x-t)+\int_0^t f(x-t+\tau,\tau)d\tau$.

Example: $\partial_t^2 u = \partial_x^2 u, u|_{t=0} = g(x), \partial_t u|_{t=0} = h(x)$. Denote $\vec{W} = (\partial_t u, \partial_x u)^T$, we have $\partial_t W = (\partial_{tt} u, \partial_{xt} u)^T = (\partial_{xx} u, \partial_{xt} u)^T = \partial_x(\partial_x u, \partial_t u)^T \Rightarrow \partial_t W_1 = \partial_x W_2, \partial_t W_2 = \partial_x W_1 \Rightarrow \partial_t(W_1+W_2) = \partial_x(W_1+W_2), \partial_t(W_1-W_2) = -\partial_x(W_1-W_2)$. The solutions are $W_1+W_2 = (W_1^0+W_2^0)(x+t), W_1-W_2 = (W_1^0-W_2^0)(x-t)$ where $W_1^0 = \partial_t u(x,0) = h(x), W_2^0 = \partial_x u(x,0) = g'(x)$. Thus $\partial_t u = \frac{1}{2}(h(x+t)+h(x-t))+\frac{1}{2}(g'(x+t)-g'(x-t)), u(x,t) = u(x,0)+\int_0^t \partial_t u d\tau = g(x)+\frac{1}{2}\int_0^t (h(x+\tau)+h(x-\tau))d\tau+\frac{1}{2}\int_0^t g'(x+\tau)-g'(x-\tau)d\tau = \frac{1}{2}\int_{x-t}^{x+t} h(s)ds+\frac{1}{2}(g(x+t)+g(x-t))$ (D'Alembert formula).

Example: $\partial_t\rho+\rho\rho_x = 0$. $p(t) = \rho(Z(t),t), \dot{p}(t) = \partial_x\rho(\dot{Z}(t)-p(t)) = 0 \Rightarrow \dot{Z}(t) = p(t), \dot{p}(t) = 0 \Rightarrow p(t) = p(0) = \rho_0(Z(0)), Z(t) = \rho_0(Z(0))t+Z(0)$. If $\rho_0(x) = x$, then $Z(t) = Z(0)t+Z(0), Z(0) = \frac{Z(t)}{1+t}, p(t) = \rho_0(Z(0)) = Z(0) = \frac{Z(t)}{1+t} \Rightarrow \rho(x,t) = \frac{x}{1+t}$. If $\rho_0(x) = -x$, then $Z(t) = -Z(0)t+Z(0), Z(0) = \frac{Z(t)}{1-t}, p(t) = -Z(0) = -\frac{Z(t)}{1-t} \Rightarrow \rho(x,t) = \frac{x}{t-1}$. We find when $t = 1$, shock happens since characteristic lines intersect.

Shock solution: recall conversation law $\int_a^b \partial_t\rho+\partial_x J(\rho)dx = 0 \Leftrightarrow \frac{d}{dt}\int_a^b \rho(x,t)dx = J(\rho(a,t))-J(\rho(b,t))$. Denote shock solution as $S(x)$, $J(\rho(a,t))-J(\rho(b,t)) = \frac{d}{dt}(\int_a^{S(t)} \rho(x,t)dx+\int_{S(t)}^b \rho(x,t)dx) = \int_a^{S(t)} \partial_t\rho dx+\rho(S(t)^-,t)\dot{S}(t)+\int_{S(t)}^b \partial_t\rho dx-\rho(S(t)^+,t)\dot{S}(t) = J(\rho(a,t))-J(\rho(S(t)^-,t))+J(\rho(S(t)^+,t)) - J(\rho(b,t))+(\rho(S(t)^-,t)-\rho(S(t)^+,t))\dot{S}(t)$. Thus $J(\rho(S(t)^-,t))-J(\rho(S(t)^+,t)) = (\rho(S(t)^-,t)-\rho(S(t)^+,t))\dot{S}(t) \Rightarrow \dot{S}(t) = \frac{J(\rho(S(t)^-,t))-J(\rho(S(t)^+,t))}{\rho(S(t)^-,t)-\rho(S(t)^+,t)}$.

4.2 Common Cases (Lagrange's Method)

Consider $a(x,y)\partial_x u+b(x,y)\partial_y u = c(x,y) \Rightarrow (a,b,c)\cdot(\partial_x u, \partial_y u, -1) = 0$. Denote curved surface $S = \{z = u(x,y)|x,y \in R\}$ in R^3, then (a,b,c) is the tangent direction of S. So denote characteristic curve as $(x(s),y(s),z(s))$, we have $\dot{x}(s) = a(x(s),y(s)), \dot{y}(s) = b(x(s),y(s)), \dot{z}(s) = c(x(s),y(s))$. A special case is $\partial_t\rho+\partial_x\rho = 0 \Rightarrow \dot{x}(s) = 1, \dot{t}(s) = 1, \dot{z}(s) = 0 \Rightarrow x(s) = x(0)+s, t(s) = t(0)+s, z(s) = z(0) \Rightarrow z(s) = \rho(x(s),t(s)) = \rho(x(0),t(0)) = \rho_0(x(0)) = \rho_0(x(s)-s) = \rho_0(x(s)-t(s))$.

How to impose IC condition? WLOG, let IC be $u(x,y)|_\Gamma = f, \Gamma = (\gamma_1(r),\gamma_2(r))$. Since we can't impose IC in the characteristic line which is in the tangent direction, so $(a,b)\cdot(-\gamma_2'(r),\gamma_1'(r)) \neq 0$ (non-characteristic condition). Then $\frac{d}{ds}x(r,s) = a(x,y), \frac{d}{ds}y(r,s) = b(x,y), \frac{d}{ds}z(r,s) = c(x,y)$ and we will get $z(r,s)$ and transform it into $z(H(x,y)) = u(x,y)$. Recall our characteristic line starts from $(r_0,0,f(r_0))$. Back to original process, $J|_\Gamma = \frac{\partial(x,y)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ y_r & y_s \end{pmatrix} = \det\begin{pmatrix} \gamma_1'(r) & a \\ \gamma_2'(r) & b \end{pmatrix} \neq 0$, so $z(r,s) = z(H(x,y))$ is feasible (local existence).

Consider common case $F(x,y,u,\partial_x u,\partial_y u) = 0$. Denote $x = x(r,s), y = y(r,s), z = z(r,s) = u(x(r,s),y(r,s)), p(r,s) = \partial_x u|_{(x(r,s),y(r,s))}, q = \partial_y u|_{(x(r,s),y(r,s))}$ where $x(r,0) = \gamma_1(r), y(r,0) = \gamma_2(r)$. Then $0 = \frac{dF}{dx} = \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial p}\frac{\partial^2 u}{\partial x^2}+\frac{\partial F}{\partial q}\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}p_x+\frac{\partial F}{\partial q}p_y, 0 = \frac{dF}{dy} = \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}q_x+\frac{\partial F}{\partial q}q_y, \frac{\partial p}{\partial s} = \frac{\partial}{\partial s}(\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial x^2}\dot{x}+\frac{\partial^2 u}{\partial x\partial y}\dot{y} = p_x\dot{x}+p_y\dot{y} = -\frac{\partial F}{\partial x}-\frac{\partial F}{\partial z}p, \frac{\partial q}{\partial s} = \frac{\partial}{\partial s}(\frac{\partial u}{\partial y}) = q_x\dot{x}+q_y\dot{y} = -\frac{\partial F}{\partial y}-\frac{\partial F}{\partial z}q$ (the last two steps are because of the following definitions: $\frac{d}{ds}x(r,s) := \frac{\partial F}{\partial p}, \frac{d}{ds}y(r,s) := \frac{\partial F}{\partial q}$). Principle: 2nd order derivative should not appear in 1st order PDE. Then $\frac{\partial z}{\partial s} = \frac{\partial u}{\partial x}\dot{x}+\frac{\partial u}{\partial y}\dot{y} = p\dot{x}+q\dot{y} = p\frac{\partial F}{\partial p}+q\frac{\partial F}{\partial q}$. This is called Lagrange's method. Next, search for initial value. $x(r,0)=\gamma_1(r), y(r,0)=\gamma_2(r), z(r,0)=f(r)$. Denote $p(r,0)=\psi_1(r), q(r,0)=\psi_2(r)$, so $F(\gamma_1(r),\gamma_2(r),f(r),\psi_1(r),\psi_2(r))=0$ and $\frac{\partial u}{\partial x}\frac{d}{dr}x(r,0)+\frac{\partial u}{\partial y}\frac{d}{dr}y(r,0)|_\Gamma=f'(r)\Leftrightarrow\psi_1(r)\gamma_1'(r)+\psi_2(r)\gamma_2'(r)=f'(r)$. Then we can get $\psi_1(r),\psi_2(r)$.

Example: $|\nabla u|=1$ when $x^2+y^2<1$ and $u=0$ when $x^2+y^2=1$ (Eikonal EQ). We have $|\nabla u|^2=1, F=p^2+q^2-1, \dot{x}_s=2p, \dot{y}_s=2q, \dot{p}_s=0, \dot{q}_s=0, \dot{z}_s=2p^2+2q^2$. IC: $x(r,0)=\cos r, y(r,0)=\sin r, \psi_1^2+\psi_2^2=1, -\psi_1\sin r+\psi_2\cos r=0$. We can get $\psi_1=\pm\cos r, \psi_2=\pm\sin r$, so $x(r,s)=\pm(2s+1)\cos r, y(r,s)=\pm(2s+1)\sin r, z=2s\Rightarrow x^2+y^2=(2s+1)^2, u(x,y)=-1+\sqrt{x^2+y^2}$ or $1-\sqrt{x^2+y^2}$.

Example: $\partial_t u=\partial_x^2 u$. Denote $v=\partial_x u$, then $\begin{pmatrix}1&0\\0&0\end{pmatrix}\partial_t\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}0&-1\\1&0\end{pmatrix}\partial_x\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$, namely $B\partial_t\vec{u}+A\partial_x\vec{u}=C\vec{u}=b(\vec{u})$. If there exists γ s.t.$\gamma^TB\partial_t\vec{u}+\gamma^TA\partial_x\vec{u}=\gamma^Tb\Leftrightarrow m^T(\alpha\partial_t\vec{u}+\beta\partial_x\vec{u})=\gamma^Tb$, then characteristic lines satisfy $\dot{t}(s)=\alpha, \dot{x}(s)=\beta$ and $m^T\dot{z}(s)=\gamma^Tb(t(s),x(s))=f(s)$. Backwards, $\gamma^TB=\alpha m^T, \gamma^TA=\beta m^T\Rightarrow\gamma^T(\beta B-\alpha A)=0\Rightarrow\det(\beta B-\alpha A)=0$. Here we find $\alpha=0$, which leads $\frac{dx}{dt}=\infty$, so method of characteristic line is unvalid.

Remark: $\forall$ PDE, movement constrained in characteristic lines satsify Newton's law of motion. For example, Burgers Equation $\partial_t u+u\partial_x u=0$, if $\dot{\phi}(t)=u(x(t),t)$, then $\ddot{\phi}(t)=0$.

5 Fourier Transform

5.1 Fourier Series

Frourier Trans: Given $f(x)\to\hat{f}(\xi)$, where $\hat{f}(\xi)=\int_{R^d}f(x)e^{-i2\pi x\cdot\xi}dx$. $f(x)\in$ real space, $\xi\in$ phase space. $\widehat{f'(x)}(\xi)=\int_{R^d}f'(x)e^{-i2\pi x\cdot\xi}dx=\int_{R^d}e^{-i2\pi x\cdot\xi}df(x)=i2\pi\xi\int_{R^d}f(x)e^{-i2\pi x\cdot\xi}dx=i2\pi\xi\hat{f}$.

Partial Fourier Trans: $\hat{u}(\xi,t)=\int_{R^d}u(x,t)e^{-i2\pi x\cdot\xi}dx$. Given $\partial_t u+\partial_x u=0$, then $\widehat{\partial_t u}+\widehat{\partial_x u}=0\Rightarrow\partial_t\hat{u}+i2\pi\xi\hat{u}=0$, which lets PDE $\to$ Algebraic EQ $\to$ ODEs.

Fourier inversion: $u(x,t)=\int_{R^d}\hat{u}(\xi,t)e^{i2\pi x\cdot\xi}d\xi$.

$f(x)\in R([-\pi,\pi])$, $f(x)\sim\sum_{n\in Z}C_n(f)e^{-inx}$ where $C_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{iny}dy$. Then $f(x)\sim C_0(f)+\sum_{n=1}^{\infty}C_n(f)e^{-inx}+\sum_{n=-1}^{-\infty}C_n(f)e^{-inx}=C_0(f)+\frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}f(y)(e^{in(y-x)}+e^{-in(y-x)})dy=C_0(f)+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}f(y)(\cos ny\cos nx+\sin ny\sin nx)dy=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$. Thus we have $a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\cos ky dy, b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\sin ky dy$.

1) Parsaval identity: $||f||_{L^2}=||\{C_n(f)\}||_{l^2}$. In fact, we have $||f-S_Nf||_{L^2}\to 0$ with $N\to+\infty$ and $S_Nf(x)\to f(x)$ a.e. What's more, if $f\to f^{(k)}\in L_1$, $C_n(f)\sim\frac{1}{n^k}$. If $f(x)$ is Lip, $C_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-int}dt=\frac{1}{2\pi}\int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}}f(t)e^{-int}dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(s+\frac{\pi}{n})e^{-in(s+\frac{\pi}{n})}ds=-\frac{1}{2\pi}f(s+\frac{\pi}{n})e^{-ins}ds\Rightarrow 2C_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t)-f(t+\frac{\pi}{n}))e^{-int}dt\Rightarrow 2|C_n(f)|\leq\frac{1}{2\pi}\text{Lip}(f)\frac{\pi}{|n|}\cdot 2\pi\Rightarrow|C_n(f)|\leq\frac{\pi}{2|n|}\text{Lip}(f)$.

2) R-L lemma: $\forall f\in L_1$, $C_n(f)\to 0$.

Gibbs phenomenon: Consider $f(x)=\frac{\pi-x}{2}, 0<x<2\pi; f(x)=0, x=0$. $S_N(f)=\sum_{n=1}^{N}\frac{\sin nx}{n}$, $\max_{0<x<\frac{\pi}{N}}S_N(f)(x)=\max_{0<x<\frac{\pi}{N}}\sum_{n=1}^{N}\frac{\sin nx}{nx}x\to\int_0^{\pi}\frac{\sin t}{t}dt\approx 0.59\pi$, which increases by about 0.09π compared to the original function.

Example: $\triangle u=0$ in $B(0,1)$, $u=g$ on $\partial B(0,1)$. Denote $u(1,\theta)=g(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$ and $u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(\phi)d\phi+\frac{1}{\pi}\sum_{n=1}^{\infty}r^n\int_{-\pi}^{\pi}g(\phi)\cos[n(\phi-\theta)]d\phi=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(\phi)d\phi+\frac{1}{\pi}\int_{-\pi}^{\pi}g(\phi)\sum_{n=1}^{\infty}r^n\cos[n(\phi-\theta)]d\phi\Rightarrow u(r,\theta)=\frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{g(\phi)}{1-2r\cos(\phi-\theta)+r^2}d\phi$.

5.2 Wave Solution

1) Standing Wave: $u(x,t) = Z(x)T(t)$. 2) Travelling wave: $u(x,t) = Z(x-ct)$.

Consider $\Box u = 0, (x,t) \in (0,\pi) \times (0,\infty), u(0,t) = u(\pi,t) = 0, u(x,0) = f(x), \partial_x u(x,t) = 0$ where $\Box = \partial_t^2 - \partial_x^2$. ANSATZ $u(x,t) = Z(x)T(t)$, then $Z(x)\ddot{T}(t) = \ddot{Z}(x)T(t) \Leftrightarrow \frac{\ddot{T}(t)}{T(t)} = \frac{\ddot{Z}(x)}{Z(x)} = \lambda =$ Const. BC: $Z(0)T(t) = Z(\pi)T(t) = 0$. So $\ddot{Z}(x) = \lambda Z(x), Z(0) = Z(\pi) = 0 \Rightarrow Z(x) = e^{\mu x}$ where $\mu^2 = \lambda$. 1) $\lambda = 0 \Rightarrow \mu = 0 \Rightarrow Z(x) = \text{Const.} \equiv 0$. 2) $\lambda > 0 \Rightarrow \mu = \pm\sqrt{\lambda} \Rightarrow Z(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$. Consider BC, $c_1 + c_2 = 0, c_1 e^{\sqrt{\lambda}\pi} + c_2 e^{-\sqrt{\lambda}\pi} = 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow Z(x) = \text{Const.} \equiv 0$. 3) $\lambda < 0 \Rightarrow \mu = \pm\sqrt{|\lambda|}i \Rightarrow Z(x) = c_1 e^{\sqrt{|\lambda|}ix} + c_2 e^{-\sqrt{|\lambda|}ix}$. Consider BC, $c_1 + c_2 = 0, c_1 e^{\sqrt{|\lambda|}i\pi} + c_2 e^{-\sqrt{|\lambda|}i\pi} = 0 \Rightarrow 2c_1 \sin(\sqrt{|\lambda|}\pi) = 0$, so $\lambda = -k^2, k = 1, 2, \cdots$, $Z_k(x) = \sin(kx)$. Then $\frac{\ddot{T}(t)}{T(t)} = \lambda = -k^2 \Rightarrow T(t) = c_1 \cos kt + c_2 \sin kt$. Thus $u_k(x,t) = (c_1 \cos kt + c_2 \sin kt)\sin kx, u(x,t) = \sum_{k=1}^{\infty}(c_k \cos kt + d_k \sin kt)\sin kx$. Consider IC, $\sum_{k=1}^{\infty} c_k \sin kx = f(x), \sum_{k=1}^{\infty} k d_k \sin kx = 0 \Rightarrow c_l = \frac{2}{\pi}\int_0^{\pi} f(x)\sin lx dx, d_l = 0, \forall l \geq 1$. So $u(x,t) = \sum_{k=1}^{\infty} c_k \cos kt \sin kx$.

Denote $f(x) = \sum_{k=1}^{\infty} c_k \sin kx$, then $u(x,t) = \frac{1}{2}\sum_{k=1}^{\infty} c_k(\sin k(x+t) + \sin k(x-t)) = \frac{1}{2}(f(x+t) + f(x-t))$ which is superposition of two waves. This usually leads weak solu.

5.3 Fourier Transform

Recall $\hat{f}(\xi) = \int_{R^d} f(x)e^{-i2\pi x\cdot\xi}dx, f(x) = \int_{R^d}\hat{f}(\xi)e^{i2\pi x\cdot\xi}d\xi$.

Gauss function: $\widehat{e^{-\pi|x|^2}} = e^{-\pi|\xi|^2}$. $\hat{f}(\xi) = \int_{R^d} e^{-\pi|x|^2}e^{-i2\pi x\cdot\xi}dx = \prod_{j=1}^{d}\int_R e^{-\pi x_j^2}e^{-i2\pi x_j\xi_j}dx_j = \prod_{j=1}^{d}\widehat{e^{-\pi x^2}}$. Denote $F(\xi) = \int_R e^{-\pi x^2}e^{-i2\pi x\xi}dx$, then $F'(\xi) = -2\pi\xi F(\xi) \Rightarrow F(\xi) = Ce^{-\pi\xi^2}$ where $C = F(0) = \int_R e^{-\pi x^2}dx = 1$. So we get the conclusion.

Translation: $x \to x+h$. $\widehat{f(x+h)}(\xi) = e^{i2\pi h\cdot\xi}\hat{f}$.

Stertching: $x \to \lambda x$. $\widehat{f(\lambda x)}(\xi) = \lambda^{-d}\hat{f}(\xi/\lambda)$.

Derivative: $\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i2\pi\xi_j\hat{f}$. To go further, $p(D) = \sum_k a_k D^k$, then $\widehat{p(D)f} = p(2\pi i\xi)\hat{f}$.

$f(x)$ is radial function $\Leftrightarrow \hat{f}$ is radial. We call a function radial iff $f(x) = f_0(|x|)$ or $f(Qx) = f(x)$ where Q is an arbitrary rotation.

$f(x)$ is a homogeneous function of order α, then $\hat{f}$ is a homo-func of order $-(d+\alpha)$.

Convolution: $\widehat{f*g} = \hat{f}\cdot\hat{g}$.

δ function: $\hat{\delta} = 1$ where $\delta(0) = 1$, $\delta(x) = 0\ \forall x \neq 0$ and $(\delta, f) = f(0)$. $1^{\vee} = \delta(x)$.

Example: $(xf(x))\hat{} = \int_R xf(x)e^{-i2\pi x\cdot\xi}dx = \int_R \frac{1}{-i2\pi}f(x)\frac{\partial}{\partial\xi}(e^{-i2\pi x\cdot\xi})dx = \frac{i}{2\pi}\frac{d}{d\xi}\int_R f(x)e^{-i2\pi x\cdot\xi}dx = \frac{i}{2\pi}(\hat{f}(\xi))'$. So $(\frac{x^2}{(1+x^2)^2})\hat{} = (xf(x))\hat{} = \frac{i}{2\pi}(\hat{f}(\xi))'$ where $\hat{f}(x) = (\frac{x}{(1+x^2)^2})\hat{} = -\frac{1}{2}[((1+x^2)^{-1})']\hat{} = -\frac{1}{2}i2\pi\xi[(1+x^2)^{-1}]\hat{} = -\frac{1}{2}i2\pi\xi\pi e^{-2\pi|\xi|} = -i\pi^2\xi e^{-2\pi|\xi|}$.

Example: $\int_{R^d}\hat{f}(\xi)e^{i2\pi x\cdot\xi}d\xi = \int_{R^d}(\int_{R^d} f(y)e^{-i2\pi y\cdot\xi}dy)e^{i2\pi x\cdot\xi}d\xi = \int_{R^d} f(y)\int_{R^d} e^{i2\pi(x-y)\cdot\xi}d\xi dy = \int_{R^d} f(y)\delta(x-y)dy = f(x)$.

Example: $\partial_t u + \partial_x u = 0, u|_{t=0} = u_0(x)$. $\widehat{\partial_t u} + \widehat{\partial_x u} = 0 \Rightarrow \partial_t\hat{u} + i2\pi\xi\hat{u} = 0, \hat{u}|_{t=0} = \hat{u}_0(\xi) \Rightarrow \hat{u}(\xi,t) = e^{-i2\pi\xi t}\hat{u}_0(\xi)$. Then $u(x,t) = \int_R \hat{u}_0(\xi)e^{-i2\pi\xi t}e^{i2\pi\xi x}d\xi = u_0(x-t)$.

Example: $-\triangle G = \delta, x \in R^3$. $\widehat{-\triangle G} = \hat{\delta} \Rightarrow 4\pi^2|\xi|^2\hat{G} = 1 \Rightarrow \hat{G} = \frac{1}{4\pi^2|\xi|^2}$. Notice $\hat{G}$ is radial $\to G$ is radial; G is a homo-func of order $-(\lambda+d) = -3-(-2) = -1$. So $G(x) = G(\frac{x}{|x|}|x|) = |x|^{-1}G(\frac{x}{|x|}) = |x|^{-1}G_0(|\frac{x}{|x|}|) = |x|^{-1}G_0(1) = C|x|^{-1}$. On the other hand, $\int_{B(0,\epsilon)} -\triangle G dx = \int_{B(0,\epsilon)}\delta(x)dx = 1 \Rightarrow -\int_{\partial B(0,\epsilon)}\frac{\partial G}{\partial n}d\sigma = 1 = -\int_{\partial B(0,\epsilon)} G_0'(\epsilon)d\sigma = -4\pi\epsilon^2 G_0'(\epsilon) \Rightarrow C = \frac{1}{4\pi}$. So $G(x) = \frac{1}{4\pi|x|}$.

Recall Parsaval idenetity: $||f||_{L^2} = ||C_n(f)||_{l_2}$.

Plancherel's Thm: $\int |f|^2 dx = \int |\hat{f}|^2 d\xi$, i.e.$||f||_{\mathscr{L}_2} = ||\hat{f}||_{\mathscr{L}_2}$.

Proof: $\int f\bar{f}dx = \int_{R^d}\int_{R^d}\hat{f}(\xi)e^{i2\pi x\cdot\xi}d\xi\overline{\int_{R^d}\hat{f}(y)e^{i2\pi x\cdot y}dy}dx = \int_{R^{2d}}\hat{f}(\xi)\overline{\hat{f}(y)}\int_{R^d}e^{i2\pi x\cdot(\xi-y)}dxd\xi dy$ $= \int_{R^d}\hat{f}(\xi)\overline{\hat{f}(\xi)}d\xi = \int_{R^d}|\hat{f}(\xi)|^2 d\xi$.

Corollary: $\int f(x)\overline{g(x)}dx = \sum_n C_n(f)\overline{C_n(g)}, \int f(x)\overline{g(x)}dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi$.

Example: $\square u = 0, u|_{t=0} = f(x), \partial_t u|_{t=0} = g(x)$. By fourier transform, we get $\partial_t^2\hat{u} + 4\pi^2\xi^2\hat{u} = 0, \hat{u}|_{t=0} = \hat{f}(\xi), \partial_t\hat{u}|_{t=0} = \hat{g}(\xi) \Rightarrow \hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi\xi t) + \hat{g}(\xi)\frac{\sin(2\pi\xi t)}{2\pi\xi}$. By fourier inversion, we get $(\hat{f}(\xi)\cos(2\pi\xi t))^{\vee} = \frac{1}{2}(f(x+t)+f(x-t)), (\hat{g}(\xi)\frac{\sin(2\pi\xi t)}{2\pi\xi})^{\vee} = \int_{R^d}\hat{g}(\xi)\frac{e^{i2\pi\xi(x+t)}-e^{i2\pi\xi(x-t)}}{4i\pi\xi}d\xi = \frac{1}{2}\int_{R^d}\hat{g}(\xi)\int_{x-t}^{x+t}e^{i2\pi\xi s}dsd\xi = \frac{1}{2}\int_{x-t}^{x+t}\int_{R^d}\hat{g}(\xi)e^{i2\pi\xi s}d\xi ds = \frac{1}{2}\int_{x-t}^{x+t}g(s)ds$.

Denote $K(t) = \frac{1}{2}\int|\partial_t u|^2 dx, P(t) = \frac{1}{2}\int|\partial_x u|^2 dx$, then $K(t)+P(t) = \frac{1}{2}\int(|\partial_t u|^2 + |\partial_x u|^2)dx = \frac{1}{2}\int(|\widehat{\partial_t u}|^2 + |\widehat{\partial_x u}|^2)dx = \frac{1}{2}\int|\partial_t\hat{u}|^2 + 4\pi^2\xi^2|\hat{u}|^2 dx$. $\partial_t\hat{u} = -2\pi\xi\hat{f}(\xi)\sin(2\pi\xi t) + \hat{g}(\xi)\cos(2\pi\xi t), 2\pi\xi\hat{u} = 2\pi\xi\hat{f}(\xi)\cos(2\pi\xi t) + \hat{g}(\xi)\sin(2\pi\xi t)$. Thus $K(t)+P(t) = \frac{1}{2}\int(4\pi^2\xi^2|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2)d\xi = \frac{1}{2}\int(|\partial_x f(x)|^2 + |g(x)|^2)dx = K(0)+P(0) \rightarrow$ conservation of energy.

$\lim_{t\to\infty}K(t) = \lim_{t\to\infty}\frac{1}{2}\int_R 4\pi^2\xi^2|\hat{f}(\xi)|^2\sin^2(2\pi\xi t) - 2\pi\xi\hat{f}(\xi)\hat{g}(\xi)\sin(4\pi\xi t) + |\hat{g}(\xi)|^2\cos^2(2\pi\xi t)d\xi$ $= \frac{1}{4}\int(4\pi^2\xi^2|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2)d\xi = \frac{1}{2}(K(0)+P(0))$. And the same holds true for $\lim_{t\to\infty}P(t)$.

Heat Equation: $\partial_t H = \triangle H, H|_{t=0} = \delta(x)$. By fourier trans, we get $\partial_t\hat{H} = -4\pi^2|\xi|^2\hat{H}, \hat{H}|_{t=0} = 1 \Rightarrow \hat{H} = e^{-4\pi^2|\xi|^2 t} \Rightarrow H = (4\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$ (Heat Kernel).

Bessel potential: $-\triangle B + B = \delta$. By fourier trans, $\hat{B} = \frac{1}{1+4\pi^2|\xi|^2} \Rightarrow B = \int_{R^d}\frac{1}{1+4\pi^2|\xi|^2}e^{i2\pi x\cdot\xi}d\xi = \int_{R^d}(\int_0^\infty e^{-(1+4\pi^2|\xi|^2)s}ds)e^{i2\pi x\cdot\xi}d\xi = \int_0^\infty e^{-s}\int_{R^d}e^{-4\pi^2|\xi|^2 s}e^{i2\pi x\cdot\xi}d\xi ds = \int_0^\infty e^{-s}(4\pi s)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4s}}ds$.

Laplace Equation: $\triangle u = 0, u(x,0) = f(x), (x,y)\in R\times R_+$. $\hat{u}(\xi,y) = \int_R u(x,y)e^{-i2\pi x\cdot\xi}dx, \partial_y^2\hat{u} - 4\pi^2\xi^2\hat{u} = 0, \hat{u}|_{(\xi,0)} = \hat{f}, \hat{u} = c_1 e^{-2\pi|\xi|y}\hat{f}(\xi) + c_2 e^{2\pi|\xi|y}\hat{f}(\xi)$. We ignore the sigular term for $y>0$, and $(e^{-2\pi|\xi|y})^{\vee} = \frac{y}{\pi(x^2+y^2)} \Rightarrow u(x,y) = \int\frac{f(x-t)y}{\pi(t^2+y^2)}dt$.

We find $(e^{-2\pi|\xi|})^{\vee} = \frac{1}{\pi(x^2+1)}$, thus $\widehat{\frac{1}{1+x^2}} = \pi e^{-2\pi|\xi|}, B(x) = (\frac{1}{1+4\pi^2|\xi|^2})^{\vee} = \int_R\frac{1}{1+4\pi^2|\xi|^2}e^{i2\pi x\cdot\xi}d\xi = (\eta = 2\pi\xi) = \frac{1}{2\pi}\int_R\frac{1}{1+\eta^2}e^{ix\eta}d\eta = \frac{1}{2}e^{-|x|}$. (dim = 1)

Newton Potential: $N(x) = (\frac{1}{4\pi^2|\xi|^2})^{\vee} = \frac{1}{(d-2)\omega_{d-1}}|x|^{2-d}$. On the other hand, $\int_0^\infty H(x,t)dt(t = \frac{|x|^2}{4s}) = \int_0^\infty(\frac{\pi|x|^2}{s})^{-\frac{d}{2}}e^{-s}\frac{|x|^2}{4}s^{-2}ds = \frac{1}{4}\pi^{-\frac{d}{2}}|x|^{2-d}\int_0^\infty s^{\frac{d}{2}-2}e^{-ds}ds = \frac{1}{4}\pi^{-\frac{d}{2}}|x|^{2-d}\Gamma(\frac{d}{2}-1) = N(x)$.

Assume $f(x)$ is radial and consider polar coordinates $\begin{cases} x_1 = r\sin\theta\cos\phi \\ x_2 = r\sin\theta\sin\phi \\ x_3 = r\cos\theta \end{cases}$ or $x = r\gamma$. Then $\hat{f}(\xi) = \int_{R^3}f(x)e^{-i2\pi x\cdot\xi}d\xi = \int_0^\infty r^2\int_{S^2}f(r\gamma)e^{-i2\pi r\gamma\cdot\xi}d\sigma(\gamma)dr = \int_0^\infty r^2 f_0(r)\int_{S^2}e^{-i2\pi r\gamma\cdot\xi}d\sigma(\gamma)dr$ and $d\sigma(\gamma) = \sin\theta d\theta d\phi$. $I(\xi) = \frac{1}{4\pi}\int_{S^2}e^{-i2\pi r\gamma\cdot\xi}d\sigma(\gamma), I(\xi) = I(Q\xi)$ (Q is an arbitrary rotation) $\Rightarrow I(\xi) = \frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^\pi e^{-i2\pi r|\xi|\cos\theta}\sin\theta d\theta = \frac{1}{2}\int_0^\pi e^{-i2\pi r|\xi|\cos\theta}\sin\theta d\theta(s = 2\pi r|\xi|\cos\theta) = \frac{1}{2}\int_{-2\pi r|\xi|}^{2\pi r|\xi|}\frac{\cos s}{2\pi r|\xi|}ds \Rightarrow \hat{f}(\xi) = \int_0^\infty r^2 f_0(r)\int_{S^2}e^{-i2\pi r\gamma\cdot\xi}d\sigma(\gamma)dr = 4\pi\int_0^\infty r^2 f_0(r)\frac{\sin 2\pi r|\xi|}{2\pi r|\xi|}dr = \frac{2}{|\xi|}\int_0^\infty rf_0(r)\sin 2\pi r|\xi|dr$. If dim $= 2, \hat{f}(\xi) = \int_{R^2}f(x)e^{-i2\pi x\cdot\xi}d\xi = \int_0^\infty r\int_{S^1}f(r\gamma)e^{-i2\pi r\gamma\cdot\xi}d\sigma(\gamma)dr = \int_0^\infty rf_0(r)\int_{S^1}e^{-i2\pi r\gamma\cdot\xi}d\sigma(\gamma)dr = [\xi = (0,-|\xi|)] = \int_0^\infty 2\pi rf_0(r)\times J_0(2\pi r|\xi|)dr$ where $J_0(2\pi r|\xi|) = \int_0^{2\pi}e^{i2\pi r|\xi|\sin\theta}d\theta$.

Yukawa potential: $-\triangle B + \mu^2 B = \delta$. By fourier trans, $\hat{B} = \frac{1}{\mu^2+4\pi^2|\xi|^2}, B = \int_{R^3}\frac{1}{\mu^2+4\pi^2|\xi|^2}e^{i2\pi x\cdot\xi}d\xi$ $= \int_0^\infty\frac{4\pi r^2}{\mu^2+4\pi^2 r^2}\frac{\sin(2\pi|x|r)}{2\pi|x|r}dr = \frac{2}{|x|}\int_0^\infty\frac{r\sin(2\pi|x|r)}{\mu^2+4\pi^2r^2}dr$ (transfer to complex variable) $= \frac{e^{-\mu|x|}}{4\pi|x|}$. (dim = 3)

Denote $f(\xi) = e^{-2\pi|\xi|}$. $(e^{-2\pi|\xi|y})^{\vee} = \frac{1}{\pi}\frac{y}{x^2+y^2}$, $e^{-2\pi|\xi|} = \frac{1}{\pi}\int\frac{1}{x^2+1}e^{i2\pi\xi\cdot x}dx = \frac{1}{\pi}\int(\int_0^\infty e^{-(x^2+1)s}ds)$ $e^{i2\pi\xi\cdot x}dx = \frac{1}{\pi}\int_0^\infty e^{-s}(e^{-sx^2})^{\vee}ds = \frac{1}{\pi}\int_0^\infty e^{-s}(\frac{s}{\pi})^{-\frac{1}{2}}e^{-\frac{\pi^2|\xi|^2}{s}}ds$. $(e^{-2\pi|\xi|})^{\wedge} = \pi^{-\frac{1}{2}}\{\overbrace{\int_0^\infty s^{-\frac{1}{2}}e^{-(s+\frac{\pi^2|\xi|^2}{s})}ds}\}$

$= \pi^{-\frac{1}{2}}\int_0^\infty s^{-\frac{1}{2}}e^{-s}(e^{-\frac{\pi^2|\xi|^2}{s}})\hat{}ds = \pi^{-\frac{1}{2}}\int_0^\infty s^{-\frac{1}{2}}e^{-s}(\frac{\pi}{s})^{-\frac{d}{2}}e^{-s|x|^2}ds = \pi^{-\frac{d+1}{2}}\int_0^\infty s^{\frac{d-1}{2}}e^{-s(1+|x|^2)}ds(t = s(1+|x|^2) = \pi^{-\frac{d+1}{2}}\int_0^\infty (1+|x|^2)^{-\frac{d-1}{2}}t^{\frac{d-1}{2}}e^{-t}(1+|x|^2)^{-1}dt = \pi^{-\frac{d+1}{2}}(1+|x|^2)^{-\frac{d+1}{2}}\int_0^\infty t^{\frac{d-1}{2}}e^{-t}dt = \pi^{-\frac{d+1}{2}}(1+|x|^2)^{-\frac{d+1}{2}}\Gamma(\frac{d+1}{2}) \to$ Gauss-Weistrass kernel.

Consider $\Box u = 0, u|_{t=0} = f(x), \partial_t u|_{t=0} = g(x)$. $\partial_t^2\hat{u} + 4\pi^2|\xi|^2\hat{u} = 0, \hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$. Def $M_t(tg) = ⨍_{|y-x|=t} tg(y)d\sigma(\gamma)$, then $\widehat{M_t(tg)} = \int_{R^3}⨍_{S^2} tg(x+t\gamma)d\sigma(\gamma)e^{-i2\pi x\cdot\xi}dx = \int_{R^3}⨍_{S^2} tg(y)e^{-i2\pi(y-t\gamma)\cdot\xi}d\sigma(\gamma)dy = \int_{R^3} tg(y)e^{-i2\pi y\cdot\xi}dy⨍_{S^2} e^{i2\pi t\gamma\cdot\xi}d\sigma(\gamma) = \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$. Thus when dim $= 3$, $u(x,t) = \frac{\partial}{\partial t}(M_t(tf)) + M_t(tg)$. Also, $M_t(f) = \frac{1}{4\pi}\int_{S^2} f(x-t\gamma)d\sigma(\gamma) = \frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^\pi f(x_1 - t\sin\theta\cos\phi, x_2 - t\sin\theta\sin\phi)\sin\theta d\theta = \frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^{\frac{\pi}{2}} f(x_1 - t\sin\theta\cos\phi, x_2 - t\sin\theta\sin\phi)\sin\theta d\theta (r = \sin\theta) = \frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^1 f(x_1 - tr\cos\phi, x_2 - tr\sin\phi)\frac{r}{\sqrt{1-r^2}}dr = \frac{1}{2\pi}\int_{|y|\le 1} f(x-ty)\frac{1}{\sqrt{1-|y|^2}}dy := \widetilde{M}_t(f)$. When dim $= 2$, $u(x,t) = \frac{\partial}{\partial t}(t\widetilde{M}_t(f)) + t\widetilde{M}_t(g)$. (Method of descent)

Regard fourier transform as an operator, i.e.$Tf = \hat{f}$, then $T : \mathscr{L}_1 \to \mathscr{L}_\infty, ||T||_{\mathscr{L}_1\to\mathscr{L}_\infty} \le 1; T : \mathscr{L}_2 \to \mathscr{L}_2, ||T||_{\mathscr{L}_2\to\mathscr{L}_2} = 1$.

Riesz-Thorin Interpolation Thm: $T : \mathscr{L}_{p_1} \to \mathscr{L}_{q_1}, T : \mathscr{L}_{p_2} \to \mathscr{L}_{q_2}, \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, then $T : \mathscr{L}_p \to \mathscr{L}_q, ||T||_{\mathscr{L}_p\to\mathscr{L}_q} \le ||T||^\theta_{\mathscr{L}_{p_1}\to\mathscr{L}_{q_1}} + ||T||^{1-\theta}_{\mathscr{L}_{p_2}\to\mathscr{L}_{q_2}}$.

Hausdorff-Young's inequality: $||\hat{f}||_{\mathscr{L}^p} \le ||f||_{\mathscr{L}^q}, \frac{1}{p} + \frac{1}{q} = 1$.

$||f * g||_r \le ||f||_p||g||_q$ where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Example: $\Box u = f, u|_{t=0} = 0, \partial_t u|_{t=0} = 0$. $\partial_t\hat{u} + 4\pi^2|\xi|^2\hat{u} = \hat{f}, \hat{u}|_{t=0} = 0, \partial_t\hat{u}|_{t=0} = 0, \hat{u}(\xi,t) = \int_0^t \frac{\sin 2\pi\xi(t-s)}{2\pi\xi}\hat{f}(\xi,s)ds$. $u(x,t) = \int_0^t\int_R \frac{\sin 2\pi\xi(t-s)}{2\pi\xi}\hat{f}(\xi,s)e^{i2\pi x\cdot\xi}d\xi ds = \frac{1}{2}\int_0^t\int_R \frac{e^{i2\pi\xi(t-s)} - e^{-i2\pi\xi(t-s)}}{i2\pi\xi} e^{i2\pi x\cdot\xi}\hat{f}(\xi,s)d\xi ds = \frac{1}{2}\int_0^t\int_R\int_{x-t+s}^{x+t-s} e^{i2\pi\xi\eta}d\eta\hat{f}(\xi,s)d\xi ds = \frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s} f(x,s)dxds$.

Example: $u_{tt} = u_{xx} + u, u|_{t=0} = g, \partial_t u|_{t=0} = h$. Denote $v(x,y,t) = u(x,t)e^y$, then $\partial_t^2 v = \partial_t^2 ue^y, \partial_x^2 v = \partial_x^2 ue^y, \partial_y^2 v = ue^y \Rightarrow v_{tt} = \partial_x^2 v + \partial_y^2 v, v(x,y,0) = ge^y, \partial_t v(x,y,0) = he^y$.

Example: (Duhammel's Principle) $U_t + AU = F$. The solution is $U(x,t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds$. Then for $u_{tt} = \triangle u + f$, $\frac{\partial}{\partial t}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \Rightarrow U_t = AU + F$. Therefore, $U = \begin{pmatrix} u \\ u_t \end{pmatrix} = S(t)\begin{pmatrix} g \\ h \end{pmatrix} + \int_0^t S(t-s)\begin{pmatrix} 0 \\ f \end{pmatrix}ds$.

Kirchhoff formula(dim $= 3$): $\Box_a u = \partial_t^2 u - a^2\partial_x^2 u = f, u|_{t=0} = \phi(x), \partial_t u|_{t=0} = \psi(x)$. The solution is $u(x,t) = \frac{1}{4\pi a^2t^2}\int_{\partial B(x,at)}[\phi(y) + D\phi(y)\cdot(y-x) + t\psi(y)]dS(y) + \frac{1}{4\pi a^2}\int_{B(x,at)}\frac{f(y,t-|y-x|/a)}{|y-x|}dy$.

For dim $= 2$, $u(x,t) = \frac{1}{2\pi at}\int_{B(x,at)}\frac{[\phi(y) + D\phi(y)\cdot(y-x) + t\psi(y)]}{\sqrt{(at)^2 - |y-x|^2}}dy + \frac{1}{2\pi a}\iint_{C(x,t)}\frac{f(y,\tau)}{\sqrt{a^2(t-\tau)^2 - |y-x|^2}}dyd\tau$, where $C(x,t) = \{(y,\tau) \in R^3 : 0 \le \tau \le t, |y-x| \le a(t-\tau)\}$.

Example: $\partial_t u = \partial_x^2 u + a\partial_x u + bu$. $\partial_t\hat{u} = -4\pi^2\xi^2\hat{u} + ai2\pi\xi\hat{u} + b\hat{u}, \hat{u}(\xi,t) = \hat{u}(\xi,0)e^{-4\pi^2\xi^2t + i2\pi a\xi t + bt}$. $u(t) = \int_R e^{-4\pi^2\xi^2t + i2\pi a\xi t}e^{i2\pi x\cdot\xi}d\xi e^{bt} = \int_R e^{-4\pi^2\xi^2t}e^{i2\pi\xi(x+at)}d\xi e^{bt} = (4\pi t)^{-\frac{1}{2}}e^{-\frac{|x+at|^2}{4t} + bt}$.

Example: $\Box u = c^2u, u|_{t=0} = g(x), \partial_t u|_{t=0} = h(x)$. Denote $v(x,y,t) = u(x,t)e^{cy}$, we get $\partial_t^2 v - \triangle v = \Box v = 0$.

Example: $\partial_t u = x^2\frac{\partial^2 u}{\partial x^2} + ax\frac{\partial u}{\partial x}$. Denote $U(y,t) = u(e^{-y},t)(x = e^{-y})$, so $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial y^2} + (1-a)\frac{\partial U}{\partial y}$. Then use fourier transform.

Example: $\Box u = 0, u|_{t=0} = x^3 + y^2z, \partial_t u|_{t=0} = 0$. Consider $\Box u_1 = 0, u_1|_{t=0} = x^3, \partial_t u_1|_{t=0} = 0$ and $\Box u_2 = 0, u_2|_{t=0} = y^2z, \partial_t u_2|_{t=0} = 0$. Assume $u_1 = \phi(x,t), u_2 = \psi(y,t)z$, we get $u_1 = \frac{1}{2}((x+t)^3 + (x-t)^3), u_2 = \frac{1}{2}((y+t)^2 + (y-t)^2)z$.

6 Fundamental Solution and Green's Function

6.1 Fundamental Solution

Consider $-\triangle u = f$, we hope $u = N * f$ where N is fundamental solution and satisfies $-\triangle N = \delta$, then $N(x) = \frac{1}{(d-2)w_d}|x|^{2-d}(d > 2)$. When $d = 2$, $N(x) = -\frac{1}{2\pi}\ln|x|$; $d = 1$, $N(x) = -\frac{1}{2}|x|$.

By fourier transform, we have already known that the solution to $-\triangle N = \delta$ is a radial function. So denote $N(x) = N_0(r), r = |x|$, then $\frac{\partial N}{\partial x_i} = N_0'(r)\frac{\partial r}{\partial x_i} = N_0'(r)\frac{x_i}{r}$, $\frac{\partial^2 N}{\partial x_i^2} = N_0''(r)(\frac{x_i}{r})^2 + N_0'(r)(\frac{1}{r} - \frac{x_i^2}{r^3})$. $\triangle N = \triangle N_0 \Rightarrow rN_0''(r) + N_0'(r)(d-1) = 0 \Rightarrow (r^{d-1}N_0'(r))' = 0, N_0'(r) = \frac{c_1}{r^{d-1}}, N_0(r) = c_2 r^{2-d}(d \neq 2)$ or $c_2 \ln r(d = 2)$.

Yukawa potential: $-\triangle Y_\mu + \mu^2 Y_\mu = \delta$. $Y_\mu(x) = Y_0(|x|), |x| = r$, so $-rY_0''(r) - 2Y_0'(r) + \mu^2 r Y_0(r) = 0$. Denote $W(r) = rY_0(r)$, then $-W''(r) + \mu^2 W(r) = 0$. Here we choose $W(r) = e^{-\mu r}, Y_0(r) = \frac{c}{r}e^{-\mu r}$. To get c, we consider the integral $\int_{B(0,\epsilon)} -\triangle Y_\mu dx + \mu^2 \int_{B(0,\epsilon)} Y_\mu dx = 1 \Rightarrow -\int_{\partial B(0,\epsilon)} \frac{\partial Y_\mu}{\partial n} d\sigma(x) + \mu^2 \int_{B(0,\epsilon)} Y_\mu dx = 1 \Rightarrow \frac{c}{\epsilon^2}(1 + \mu\epsilon)e^{-\mu\epsilon}4\pi\epsilon^2 + \mu^2 \int_{B(0,\epsilon)} \frac{c}{r}e^{-\mu r}dx = 1(\epsilon \to 0) \Rightarrow c = \frac{1}{4\pi}$.

Wave equation: $\Box W = 0, W|_{t=0} = 0, \partial_t W|_{t=0} = \delta$. $\hat{W} = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}, W = \int_{R^3} \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} e^{i2\pi x\cdot\xi} d\xi = \int_0^\infty \frac{r\sin(2\pi rt)}{2\pi} \int_{S^2} e^{i2\pi x\cdot r\gamma} d\sigma(\gamma) dr = \frac{1}{2\pi|x|}\int_0^\infty \sin(2\pi rt)\sin(2\pi r|x|)dr = \frac{-1}{8\pi|x|}\int_R (e^{i2\pi r(t+|x|)} - e^{-i2\pi r(t-|x|)} - e^{-i2\pi r(|x|-t)} - e^{-i2\pi r(t+|x|)})dr = \frac{1}{4\pi|x|}\delta(t - |x|)$. So $W * h = \int_{R^3} \frac{\delta(t-|y|)}{4\pi|y|} h(x-y)dy = \frac{1}{4\pi}\int_{S^2}\int_0^\infty r\delta(t - r)h(x - r\gamma)drd\sigma(\gamma) = \frac{1}{4\pi}\int_{S^2} th(x - t\gamma)d\sigma(\gamma) = tM_t(h)$. The solution of common cases is given by $\partial_t(W * u_0) + W_t * u_1$. (All the discussions here are in the condition of $d = 3$)

Heat equation: $\partial_t H = \partial_x^2 H, H|_{t=0} = \delta(x)$. We notice that $\lambda^d H(\lambda x, \lambda^2 t)$ is a fundamental solution ($\delta(\lambda x) = \lambda^{-d}\delta(x)$), so $\lambda^d H(\lambda x, \lambda^2 t) = H(x,t)(\lambda = t^{-\frac{1}{2}}) \Rightarrow H(x,t) = t^{-d/2}h(x/t^{1/2})$(radial) $= t^{-d/2}w(|x|/t^{1/2})$. Taking it to the previous equation, we can get $\frac{1}{2}(r^d w(r))' + (r^{d-1}w'(r))' = 0 \Rightarrow \frac{r^d}{2}w(r) + r^{d-1}w'(r) = 0 \Rightarrow W(r) = Ce^{-\frac{1}{4}r^2}$.

Example: $\partial_t u + \triangle(u^\gamma) = 0, x \in R^d, \gamma > 1, u \geq 0$. We assume the solution has the form $t^{-\alpha}v(\frac{x}{t^\beta})$(self-similar), then $t^{-\alpha-1}(u + (y\cdot\nabla)v)(y) = t^{-(\alpha\gamma+2\beta)}\triangle v^\gamma$ where $y = \frac{x}{t^\beta} \Rightarrow \alpha + 1 = \alpha\gamma + 2\beta$. To find fundamental solution, we assume $v(\frac{x}{t^\beta}) = w(\frac{|x|}{t^\beta})$ and denote $r = \frac{|x|}{t^\beta}$. So $\triangle u^\gamma = t^{-\alpha\gamma}\triangle w^\gamma = t^{-\alpha\gamma-2\beta}(w^\gamma + \frac{d-1}{r}(w^\gamma)'), \partial_t u = t^{-\alpha-1}(\alpha w + \beta r w'(r)) \Rightarrow \alpha r w + \beta r^2 w'(r) + r(w^\gamma)'' + (d-1)(w^\gamma)' = 0 \Rightarrow \alpha r^{d-1}w + \beta r^{d-1}w'(r) + (r^{d-1}(w^\gamma)')'$. Assume $\alpha = d\beta$, then $(\beta r^d w + r^{d-1}(w^\gamma)')' = 0 \Rightarrow (w^\gamma)' = -\beta r w \Rightarrow \frac{\gamma}{\gamma-1}(w^{\gamma-1})' = -\beta r \Rightarrow w = (C - \frac{\beta(\gamma-1)}{2\gamma}r^2)^{\frac{1}{\gamma-1}} = (C - \frac{\beta(\gamma-1)}{2\gamma}\frac{|x|^2}{t^{2\beta}})^{\frac{1}{\gamma-1}}$.

6.2 Green's Function

Consider $\begin{cases} -\triangle u = f \text{ in } \Omega \\ u = g \quad \text{on } \partial\Omega \end{cases}$ (Dirichlet BVP). We hope we can represent the solution in terms of Green's function. $0 = \int_\Omega (\triangle u + f)vdx = -\int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma(x) + \int_\Omega f\cdot v dx \Rightarrow \int_\Omega \triangle u \cdot v dx = -\int_\Omega \nabla u\nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma(x), \int_\Omega \triangle v\cdot u dx = -\int_\Omega \nabla v \nabla u dx + \int_{\partial\Omega}\frac{\partial v}{\partial n} u d\sigma(x) \Rightarrow \int_\Omega (\triangle u \cdot v - \triangle v \cdot u)dx = \int_{\partial\Omega}(\frac{\partial u}{\partial n}v - \frac{\partial v}{\partial n}u)d\sigma(x)$. Define Green function as $\begin{cases} -\triangle G_y(x) = \delta(x-y), x \in \Omega \\ G_y(x) = 0, x \in \partial\Omega \end{cases}$, so the above can be rewritten as $-\int_\Omega f(x)G_y(x)dx + u(y) = -\int_{\partial\Omega}\frac{\partial G_y}{\partial n}g(x)d\sigma(x)$ where $G_y = v$ and then $u(y) = \int_\Omega f(x)G_y(x)dx - \int_{\partial\Omega}\frac{\partial G_y}{\partial n}g(x)d\sigma(x)$.

Example: $-u''(x) = f(x), u(0) = u(1) = 0$. Green function $-G_y''(x) = \delta(x-y), G_y(0) =$

$G_y(1) = 0$. For $x \neq y, -G_y''(x) = 0 \Rightarrow G_y(x) = \begin{cases} A(y)x, 0 < x < y \\ C(y)(x-1), y < x < 1 \end{cases}$. $\int_{y-\epsilon}^{y+\epsilon} -G_y''(x)dx = \int_{y-\epsilon}^{y+\epsilon} \delta(x-y)dx = 1 \Rightarrow -G_y'(y+\epsilon) + G_y'(y-\epsilon) = 1, -C(y) + A(y) = 1$. $\int_{y-\epsilon}^{y+\epsilon} \int_0^x G_y''(x')dx'dx = \int_{y-\epsilon}^{y+\epsilon} \int_0^x \delta(x'-y)dx'dx = \epsilon(\epsilon \to 0) \Rightarrow G_y(x) \in C[0,1] \Rightarrow A(y)y = C(y)(y-1)$. Thus $A(y) = 1-y, C(y) = -y$ and $u(y) = \int_0^1 G_y(x)f(x)dx = \int_0^1 (\min(x,y) - xy)f(x)dx$.

Consider $\begin{cases} -\triangle u = f \text{ in } \Omega \\ \partial_n u = g \text{ on } \partial\Omega \end{cases}$. Green function $\begin{cases} -\triangle G_y(x) = \delta(x-y) - \frac{1}{|\Omega|}, x \in \Omega \\ \frac{\partial G_y}{\partial n} = 0, x \in \partial\Omega \end{cases}$. $u(y) = \int_\Omega f(x)G_y(x)dx + ⨍_\Omega u dx + \int_\Omega g(x)G_y(x)d\sigma(x)$. The solution is not unique. ($-\int_\Omega f dx = \int_{\partial\Omega} g d\sigma$).

Example: $\begin{cases} -\nabla \cdot (A\nabla u) + (b \cdot \nabla)u = f \\ u = g \end{cases}$. $\int_\Omega \mathscr{L}[u]v dx = \int_\Omega (-\nabla \cdot (A^T\nabla v) + \nabla \cdot (bv))u dx + \int_{\partial\Omega} (b \cdot n)uv d\sigma(x) + \int_{\partial\Omega} (n \cdot (A^T\nabla v)u - n \cdot (A\nabla u)v)d\sigma(x)$. Define $\mathscr{L}^*[u] = -\nabla(A^T\nabla u) + \nabla \cdot (bu)$. Green function $\begin{cases} \mathscr{L}^* G_y(x) = \delta(x-y), x \in \Omega \\ G_y(x) = 0, x \in \partial\Omega \end{cases}$.

Consider $\begin{cases} -\triangle G_y(x) = \delta(x-y), x \in \Omega \\ G_y(x) = 0, x \in \partial\Omega \end{cases}$. Assume $G_y(x) = N_y(x) + H(x)$ where $N_y(x)$ is Newton potential, then $\begin{cases} -\triangle H_y(x) = 0, x \in \Omega \\ H_y(x) = -N_y(x), x \in \partial\Omega \end{cases}$.

Example: $\begin{cases} -\triangle u = 0, x \in R \times R_+ \\ u = f(x), x \in \partial R \times R_+ \end{cases}$. Method of image: $N_y(x) = -\frac{1}{2\pi}\ln|x-y|$, and $G_y(x) = N_y(x) - N_{y^*}(x)$ where $y_2 + y_2^* = 0$. Then $\frac{\partial G_y}{\partial n} = -\frac{1}{\pi}\frac{y_2}{|x-y|^2}, u(y) = \frac{1}{\pi}\int_R \frac{f(x_1)y_2}{(x_1-y_1)^2+y_2^2}dx_1$.

Example: $\begin{cases} -\triangle u = 0, x \in R_+ \times R_+ \\ u = f(x), x \in \partial R_+ \times R_+ \end{cases}$. $G_y(x) = N_y(x) - N_{y_1}(x) - N_{y_2}(x) + N_{y_3}(x)$ where $y_1 = (1,-1)y, y_2 = (-1,1)y, y_3 = (-1,-1)y$ (entry-wise).

Example: $\begin{cases} -\triangle u = f, u \in B_R(0) \\ u = g, u \in \partial B_R(0) \end{cases}$. Kelvin transform: $y^* = \frac{R^2}{|y|^2}y$. Thus $|x-y^*|^2 = |x|^2 - 2(x,y)\frac{R^2}{|y|^2} + \frac{R^4}{|y|^2}(x \in \partial B_R(0)) = \frac{R^2}{y^2}(|y|^2 - 2(x,y) + |x|^2) \Rightarrow |x-y^*| = \frac{R}{|y|}|x-y|$. Then $G_y(x) = \frac{1}{(d-2)w_{d-1}}|x-y|^{2-d} - \frac{1}{(d-2)w_{d-1}}(\frac{|y|}{R}|x-y^*|)^{2-d}, d \geq 3$ and $G_y(x) = -\frac{1}{2\pi}\ln|x-y| + \frac{1}{2\pi}\ln\frac{|y|}{R}|x-y^*|, d = 2$. $u(y) = \int_{B_R(0)} = \int_\Omega f(x)G_y(x)dx - \int_{\partial B_R(0)} g(x)\frac{\partial G_y}{\partial n}d\sigma(x)$. If $d \geq 3$, $\frac{\partial G_y}{\partial n} = \frac{x_i}{R}\frac{\partial G_y}{\partial x_i} = \frac{x_i}{R}(-\frac{1}{w_{d-1}}|x-y|^{1-d}\frac{x_i-y_i}{|x-y|} + \frac{1}{w_{d-1}}|x-y^*|^{1-d}\frac{x_i^*-y_i^*}{|x-y^*|}(\frac{|y|}{R})^{2-d}) = \frac{x_i}{R}\frac{1}{w_{d-1}}(-\frac{x_i-y_i}{|x-y|^d} + \frac{x_i-y_i^*}{|x-y^*|^d(|y|/R)^d}\frac{|y|^2}{R^2}) = \frac{-1}{w_{d-1}R|x-y|^d}(|y|^2 - R^2)$. Assume $f(x) = 0$, $u(y) = \frac{R^2-y^2}{w_{d-1}R}\int_{\partial B_R(0)} \frac{g(y)}{|x-y|^d}d\sigma(x)$, and $u(0) = ⨍_{\partial B_R(0)} u(y)d\sigma(y)$. By taking integral w.r.t. r from 0 to R, $u(0) = ⨍_{B_R(0)} u(y)dy$.

Symmetric: $\begin{cases} -\triangle G_a(x) = \delta(x-a), x \in \Omega \\ G_a(x) = 0, x \in \partial\Omega \end{cases}$, $\begin{cases} -\triangle G_b(x) = \delta(x-b), x \in \Omega \\ G_b(x) = 0, x \in \partial\Omega \end{cases}$. $G_b(a) - G_a(b) = \int_\Omega (\triangle G_a)G_b - (\triangle G_b)G_a dx = \int_{\partial\Omega} \frac{\partial G_a}{\partial n}G_b - \frac{\partial G_b}{\partial b}G_a d\sigma = 0$.

Example: $\begin{cases} -\triangle u = f, x \in B(0,R) \\ \frac{\partial u}{\partial n} = g, x \in \partial B(0,R) \end{cases}$ $(d = 2)$. Assume $G_y(x) = N_y(x) + N_{y^*}(x) = -\frac{1}{2\pi}\ln|x -$

$y| - \frac{1}{2\pi}\ln\frac{|y|}{R}|x-y^*|$. $\frac{\partial G_y}{\partial n} = -\frac{1}{2\pi R}$. Transfer the former question to $\begin{cases} -\triangle H(x) = -\frac{1}{\pi R^2} \\ \frac{\partial H}{\partial n} = \frac{1}{2\pi R} \end{cases}$. Assume $H(x) = H_0(r)$, then $H_0''(r) + \frac{1}{r}H_0'(r) = \frac{1}{\pi R^2}, H_0'(R) = \frac{1}{2\pi R} \Rightarrow H_0(r) = \frac{r^2}{4\pi R^2} + C$.

7 Fourier Method

7.1 Eigenvalue Problem and Separation of Variables

Consider $\begin{cases} -u''(x) = \lambda u \\ u(0) = u(\pi) = 0 \end{cases}$, $\lambda_k = k^2, u_k = \sin kx$. $\begin{cases} -u''(x) = \lambda u \\ u'(0) = u'(\pi) = 0 \end{cases}$, $\lambda_k = k^2, u_k = \cos kx$. $\begin{cases} -u''(x) = \lambda u \\ u(0) = 0, u'(\pi) = 0 \end{cases}$, $\lambda_k = (k+\frac{1}{2})^2, u_k = \sin(k+\frac{1}{2})x$. $\begin{cases} -u''(x) = \lambda u \\ u'(0) = 0, u(\pi) = 0 \end{cases}$, $\lambda_k = (k+\frac{1}{2})^2, u_k = \cos(k+\frac{1}{2})x$.

Example: $\begin{cases} -u''(x) = f(x) \\ u(0) = u(\pi) = 0 \end{cases}$. $\widetilde{u}_k(x) = \sqrt{\frac{2}{\pi}}\sin kx, u(x) = \sum a_k\widetilde{u}_k(x), -u''(x) = -\sum a_k\widetilde{u}_k''(x) = \sum \lambda_k a_k \widetilde{u}_k(x)$. $f(x) = \sum f_k\widetilde{u}_k(x) \Rightarrow \lambda_k a_k = f_k, u(x) = \sum \frac{f_k}{a_k}\widetilde{u}_k(x) = \sum_{k=1}^{\infty}\lambda_k^{-1}\int_0^1 f(y)\widetilde{u}_k(y)dy\widetilde{u}_k(x) = \int_0^1 f(y)\left(\sum_{k=1}^{\infty}\frac{\widetilde{u}_k(x)\widetilde{u}_k(y)}{\lambda_k}\right) = \int_0^1 f(y)G_x(y)dy$ where $G_x(y) = \sum_{k=1}^{\infty}\frac{2}{\pi}\frac{\sin kx \sin ky}{k^2} = \min(x,y) - \frac{xy}{\pi}$.

Example: $\begin{cases} \partial_t u = \triangle u \\ u(0) = u(\pi) = 0 \\ u|_{t=0} = u_0(x) \end{cases}$. Assume $u(x,t) = T(t)Z(x), Z_k(x) = \widetilde{u}_k(x)$, then $\dot{T}(t) = -\lambda_k T(t), T(t) = e^{-\lambda_k t}T(0), u(x,t) = \sum_{k=1}^{\infty}a_k e^{-k^2 t}\widetilde{u}_k(x), u(x,0) = \sum_{k=1}^{\infty}a_k\widetilde{u}_k(x) = u_0(x)$, thus $a_k = \int_0^{\pi}u_0(x)\widetilde{u}_k(x)dx$.

Example: $\begin{cases} \Box u = 0 \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = g \\ u(0,t) = u(\pi,t) = 0 \end{cases}$. Assume $u = \sum_k T_k(t)Z_k(x)$ where $Z_k(x) = \sqrt{\frac{2}{\pi}}\sin kx$, we get $u = \sum_{k=1}^{\infty}(c_k\cos kt + d_k\sin kt)Z_k(x)$ where $c_k = \int_0^{\pi}f(x)Z_k(x)dx, d_k = \frac{1}{k}\int_0^{\pi}g(x)Z_k x dx$. Denote $u_n = \sum_{k=1}^{n}(c_k\cos kt + d_k\sin kt)Z_k(x)$, then $\Box u_n = 0, 0 = \int_0^{\infty}dt\int_0^{\pi}\Box u_n\phi(x,t)dx = \int_0^{\infty}dt\int_0^{\pi}(\partial_t^2 u_n - \partial_x^2 u_n)\phi(x,t)dx$. $\int_0^{\pi}dx\int_0^{\infty}\partial_t^2 u_n\phi dt = \int_0^{\pi}\int_0^{\infty}\phi d\partial_t u_n = \int_0^{\pi}dx(-\phi(x,0)\partial_t u_n(x,0)) - \int_0^{\infty}\partial_t u_n\partial_t\phi dx = -\int_0^{\pi}\phi(x,0)g_n dx - \int_0^{\pi}\int_0^{\infty}\partial_t\phi du_n = -\int_0^{\pi}\phi(x,0)g_n dx + \int_0^{\pi}\partial_t\phi(x,0)f_n dx, \int_0^{\infty}dt\int_0^{\pi}-\partial_x^2 u_n\phi(x,t)dx = \int_0^{\pi}\int_0^{\infty}u_n\Box\phi dx dt$. Let $n\to\infty$, we get $-\int_0^{\pi}\phi(x,0)g(x)dx + \int_0^{\pi}\partial_t\phi(x,0)f(x)dx + \int_0^{\pi}\int_0^{\infty}u\Box\phi dx dt \to$ weak solu.

If we replace BC with inhomogeneous condition, i.e.$u(0,t) = A(t), u(\pi,t) = B(t)$. Denote $w(x,t) = \frac{B-A}{\pi}x + A(t)$ and $v = u - w$, we get $\Box v = -\partial_t^2 w := f(x,t), v(x,0) = f(x) - w(x,0), \partial_t v(x,0) = g(x) - \partial_t w(x,0) \to \Box v = f(x,t), v(x,0) = f_1, \partial_t v(x,0) = g_1, v(0,t) = v(\pi,t) = 0$. Then use Duhammel's principle to acquire a special solution.

Example: $-\triangle u = 0, \Omega = (0,a)\times(0,b), u(x,0) = f(x), u(x,b) = u(0,y) = u(a,y) = 0$. Assume $u(x,y) = Z(x)Y(y), \ddot{Z}(x)Y(y) + Z(x)\ddot{Y}(y) = 0, \frac{\ddot{Z}(x)}{Z(x)} = -\frac{\ddot{Y}(y)}{Y(y)}$. So $-\ddot{Z}(x) = \lambda Z(x), Z(0) = Z(a) = 0$.

Eignevalue $\lambda_k = (\frac{k\pi}{a})^2, Z_k(x) = \sqrt{\frac{2}{a}}\sin\frac{k\pi x}{a}$. $\ddot{Y}(y) = \lambda_k Y(y), Y(b) = 0, Y_k(y) = \sinh\frac{k\pi}{a}(b-y)$. Thus $u(x,y) = \sum_{k=1}^{\infty} a_k Z_k(x) Y_k(y), a_k = \frac{(f,Z_k)}{Y_k(0)} \Rightarrow u(x,y) = \sum_k f_k \frac{Y_k(y)}{Y_k(0)} Z_k(x)$. $\frac{Y_k(y)}{Y_k(0)} = \frac{\sinh\frac{k\pi}{a}(b-y)}{\sinh\frac{k\pi}{a}b} = \frac{e^{-\frac{k\pi}{a}y} - e^{-\frac{k\pi}{a}(2b-y)}}{1-e^{-\frac{2k\pi}{a}b}}$. $\lim_{y\to 0} u(x,y) = f(x)$.

Example: $-\triangle u = f, \Omega = B_R(0), u(R,\theta) = g(\theta)$. Assume $u(r,\theta) = R(r)\Theta(\theta)$, then $\ddot{R}(r)\Theta(0) + \frac{1}{r}\dot{R}(r)\Theta(0) + \frac{R(r)}{r^2}\ddot{\Theta}(\theta) = 0 \Rightarrow r^2\frac{\ddot{R}(r)}{R(r)} + r\frac{\dot{R}(r)}{R(r)} = -\frac{\ddot{\Theta}(\theta)}{\Theta(\theta)}$. Periodic BC: $\Theta(0) = \Theta(2\pi), \Theta'(0) = \Theta'(2\pi)$. Eigenvalue $\lambda_k = k^2, r^2\ddot{R}(r) + r\dot{R}(r) = -k^2 R(r)$ (Euler Equation), $R(r) = r^\alpha, \alpha^2 r^\alpha = k^2 r^\alpha, \alpha = \pm k \Rightarrow R(r) = r^k, r^{-k} (k \geq 1)$ and $R(r) = 1, \ln r (k=0)$. Thus $u = \alpha + \beta\ln r + \sum_{k=1}^{\infty} r^k(a_k\cos k\theta + b_k\sin k\theta) + \sum_{k=1}^{\infty} r^{-k}(c_k\cos k\theta + d_k\sin k\theta)$.

Example: $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. Assume $u(x,y) = Z(x)Y(y), (\dot{Z}(x)Y(y))^2\ddot{Z}(x)Y(y) + 2\dot{Z}(x)Y(y)Z(x)\dot{Y}(y)\dot{Z}(x)\dot{Y}(y) + (Z(x)\dot{Y}(y))^2 Z(x)\ddot{Y}(y) = 0$. Let $\dot{Z}(x) = \lambda_1 Z(x), \dot{Y}(y) = \lambda_2 Y(y)$, then $\lambda_1^4 Z^3Y^3 + 2\lambda_1^2\lambda_2^2 Z^3Y^3 + \lambda_2^4 Z^3Y^3 = 0 \Rightarrow \lambda_1^2 + \lambda_2^2 = 0$. So $u(x,y) = e^{\lambda_1 x}e^{\lambda_2 y} = e^{\lambda_2(y\pm ix)} (\lambda_2 < 0)$. In anthoer way, assume $u(x,y) = Z(x) + Y(y), (\dot{Z})^2\ddot{Z} + (\dot{Y}(y))^2\ddot{Y}(y) = 0, \ddot{Z}(x)(\dot{Z})^2 = \lambda, \ddot{Y}(y)(\dot{Y}(y))^2 = -\lambda \Rightarrow (\dot{Z})^3 = 3\lambda x, \dot{Z} = (3\lambda x)^{1/3}, Z = C + \frac{4}{3}(3\lambda x)^{4/3}$.

Example: $\partial_t u = \triangle(u^\gamma)$. Assume $u = T(t)Z(x), \dot{T}(t)Z(x) = T^\gamma(t)\triangle Z^\gamma, \frac{\dot{T}(t)}{T^\gamma(t)} = \frac{\triangle Z^\gamma}{Z(x)} = \lambda$. $\frac{1}{1-\gamma}T^{1-\gamma}(t) = C_1 + \lambda t, T(t) = ((1-\gamma)C_1 + (1-\gamma)\lambda t)^{\frac{1}{1-\gamma}}$. Also, $\triangle Z^\gamma = \lambda Z, Z(x) = |x|^\alpha, Z^\gamma = |x|^{\alpha\gamma}, \alpha\gamma(\alpha\gamma-1)|x|^{\alpha\gamma-2} + (d-1)\alpha\gamma|x|^{\alpha\gamma-2} = \lambda|x|^\alpha \Rightarrow \alpha\gamma - 2 = \alpha, \alpha = \frac{2}{\gamma-1}, \lambda = \alpha\gamma(\alpha\gamma + d - 2)$.

Consider $\begin{cases} \mathcal{L}u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$, $\begin{cases} \mathcal{L}u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$. Green function $\begin{cases} \mathcal{L}G_y(x) = \delta(x-y) \text{ in } \Omega \\ G_y(x) = 0 \text{ on } \partial\Omega \end{cases}$, $G_y(x) = \sum_{k=1}^{\infty} a_k(y)u_k(x), \mathcal{L}G_y(x) = \sum_{k=1}^{\infty} a_k(y)\mathcal{L}u_k(x) = \sum_{k=1}^{\infty} a_k(y)\lambda_k u_k(x) = \delta(x-y)$. So $(\delta(x-y), u_k(x)) = a_k(y)\lambda_k(u_k, u_k) \Rightarrow u_k(y) = \lambda_k a_k(y), a_k(y) = \lambda_k^{-1}u_k(y), G_y(x) = \sum_{k=1}^{\infty}\lambda_k^{-1}u_k(x)u_k(y)$.

7.2 Variation of Constant for PDE

Recall $\dot{u} = Au + f, u(0) = u_0$. The solu is $u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(s)ds$.

Consider $u_t = \triangle u + f, u|_{t=0} = u_0$. Regard $A = \triangle, e^{At}u_0 = \int_{R^d}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}u_0(y)dy$, then $u(t) = \int_{R^d}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}u_0(y)dy + \int_0^t\int_{R^d}(4\pi(t-s))^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds$.

Consider $\dot{u} = Au + f, u(0) = 0$. Denote $v(t,s)$ satisfies $v_t(t,s) = Av(t,s), v(t,t) = f(t)$, thus $v(t,s) = e^{A(t-s)}f(s)$ and $u(t) = \int_0^t v(t,s)ds$.

Consider $\partial_t u = \triangle u + f, u|_{t=0} = 0$. Denote $v(x,t,s)$ satisfies $v_t(x,t,s) = \triangle v(x,t,s), v(x,t,s)|_{s=t} = f(x,t)$. The solution is given by $u(x,t) = \int_0^t v(x,t,s)ds$.

8 Energy Method

Consider wave equation $\Box u = 0, u = g, u|_{t=0} = u_0, \partial_t u|_{t=0} = u_1, \Omega = \{(x,t) \in R^3 \times R : |x-x_0| \leq |t-t_0|\}$, and energy density $e(t) = \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2$. $\dot{e}(t) = u_t u_{tt} + \nabla u\nabla u_t = u_t\triangle u + \nabla u\nabla u_t = \nabla\cdot(u_t\nabla u)$. Denote $m(t) = u_t\nabla u$, then $e_t - \nabla\cdot m(t) = 0 = \int_F(e_t - \nabla\cdot m(t))dxdt$ where F is a frustum (i.e. a piece of solid light cone). By Gauss's formula, we obtain $\int_{\partial F}(n_t e(t) - n\cdot m(t))dx = 0$. Denoting T, B, K as the top, the bottom and the side of F, we have $\int_T e(t)dx - \int_B e(t)dx + \int_K(n_t e(t) - u_t\frac{\partial u}{\partial n})dx = 0$. The third term is equal to $\frac{1}{\sqrt{2}}\int_K(e(t) - u_t\frac{\partial u}{\partial n})dx \geq \frac{1}{\sqrt{2}}\int_K(e(t) - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2)dx = 0$, so $\int_T e(t)dx \leq \int_B e(t)dx$.